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# Random Processes in Electrical and Mechanical Systems

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V. L. Lebedev / R

Proc in Electrical and Mechanical Systems



В. Л. ЛЕБЕДЕВ

V. L. Lebedev

# СЛУЧАЙНЫЕ ПРОЦЕССЫ В ЭЛЕКТРИЧЕСКИХ И МЕХАНИЧЕСКИХ СИСТЕМАХ

SLUCHAINYE PROTSESSY V ELEKTRICHESKIKH  
I MEKHANICHESKIKH SISTEМАХ

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ФИЗИКО-МАТЕМАТИЧЕСКОЙ ЛИТЕРАТУРЫ

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## CONTENTS

Preface	1
Chapter One. GENERAL CONCEPTS OF RANDOM PROCESSES	3
§ 1. Dynamical and statistical laws	3
§ 2. Random processes	5
§ 3. The role of the theory of random processes in engineering	6
Chapter Two. RANDOM FUNCTIONS AND THEIR LINEAR TRANSFORMATIONS	8
§ 4. The concept of random function	8
§ 5. Stationary and nonstationary random functions	9
§ 6. Moments	9
§ 7. The derivative of a random function	14
§ 8. Integral transformations of random functions	18
§ 9. A set of several integral transformations of the random function	20
§10. Linear transformations of random functions	21
Chapter Three. RANDOM FORCES UPON A LINEAR SYSTEM WITH LUMPED CONSTANTS	24
§11. Statement of the problem and terminology	24
§12. The method of stochastic differential equations	25
§13. The method of impulse characteristics	28
§14. The spectral method	32
§15. An RC circuit excited by a stationary fluctuating voltage	42
§16. Uncorrelated input	47
§17. The problem of two RC-circuits with a common input	50
§18. The problem of two RC-circuits with a common output	53
Chapter Four. SOME LINEAR PROBLEMS IN THE THEORY OF RANDOM PROCESSES	57
§19. One-dimensional Brownian motion	57
§20. Thermal noise in electric circuits	59
§21. Thermal noise in an electrical oscillation circuit	61
§22. Thermal motion of a galvanometer	64
§23. The passage of irregular telegraph signals through a linear filter	69
§24. The optimal filter problem	73
§25. Elements of the theory of potential-noise stability	79

Chapter Five.	RANDOM FORCE ON A NONLINEAR SYSTEM	83
§ 26.	Simplest problem of random force on an inertialess nonlinear system	83
§ 27.	The general problem of random force on an inertialess nonlinear system	87
§ 28.	Random process in inertial nonlinear systems	91
Chapter Six.	SOME NONLINEAR PROBLEMS IN THE THEORY OF RANDOM PROCESSES	93
§ 29.	Action of fluctuation noise on a detector with exponential characteristic	93
§ 30.	Statistical properties of the noise voltage envelope at the output of a selective system	96
§ 31.	Statistical properties of the phase of a noise voltage at the output of a selective system	104
§ 32.	Statistical dependence between envelopes of noise voltages at the outputs of two selective four-poles with their inputs connected in parallel	108
§ 33.	Statistical properties of the voltage envelope at the output of a selective system under action of non-modulated signal and fluctuation noise	112
§ 34.	Statistical properties of the voltage phase at the output of a selective system under action of a nonmodulated signal and fluctuation noise	115
§ 35.	Computation of the detector output for given statistical properties of the applied voltage envelope	116
Bibliography		119

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#### Note

This book deals with stationary and nonstationary random processes in linear systems and in inertialess nonlinear systems which have either one or several inputs and outputs. The application of the mathematical methods expounded is illustrated by many practical examples related to various physical and engineering problems.

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## PREFACE

At present a wide range of problems related to various branches of physics and engineering are solved by probability methods. Naturally, no detailed exposition of all the multitudinous applications of probability theory and mathematical statistics to various physical and engineering problems can be included within the scope of one book. For the small monograph hereby presented to the reader a much smaller range was chosen: the theory of electrical and mechanical systems with lumped constants under a statistically characterized external force.

The material is arranged in the following manner. After the first chapter which is of introductory nature, the principles of the theory of random functions and their linear transformations are given in brief. This mathematical material, subsequently required, constitutes the second chapter.

On the basis of relationships obtained in Chapter Two, the third chapter deals with the general methods of analysis of random processes in linear systems with lumped constants. Beside the generally accepted spectral method, the method of stochastic differential equations and the method of impulse characteristics are described here in sufficient detail.

The equations given in this chapter for each of the three methods mentioned are applicable not only to systems with one input and one output, but also to the most general problem of random action upon a linear system which has several inputs and outputs. Several problems are then examined by way of illustration. These problems are solved by each of the three methods, thus affording their comparison.

In Chapter Four the application of the methods of Chapter Three to a wider range of problems is shown (the Brownian movement, electric fluctuations, thermal movement of a galvanometer, transmission of telegraph signals and fluctuation noise through a low-pass filter, elements of the theory of optimal systems). The results obtained here should in no case be considered as exhaustive. The author has only desired to show with the aid of these examples the diversity of possible problems relating to random action upon a linear system and to further exemplify the general methods.

Chapter Five contains mainly the theory of inertialess nonlinear systems under random action, for which such general relationships have been obtained which make it possible to carry out calculations in all cases met with in practice. The hitherto little-developed theory of random processes in inertial nonlinear systems is briefly surveyed in a separate section.

In Chapter Six a series of practical examples is given which illustrate the methods of Chapter Five.

In the course of writing this book, the author has endeavored to confine himself as far as possible to the higher mathematics course given in technical institutes. Only elementary notions of the theory of probability and operational calculus are

required of the reader. These notions can, if necessary, be easily obtained from existing specialized literature. The author has consciously allowed some instances of lack of mathematical rigor in computations. Refraining from excessive mathematical rigor makes it possible to substantially simplify the exposition and, at the same time, in the overwhelming majority of practical cases, it does not impose any restriction on the applicability of results obtained. The author assumes that some readers will easily notice and supply these deficiencies in rigor while others will readily accept them.

In conclusion the author considers it his pleasant duty to express his gratitude to the critics Prof. S. M. Rytov and Dr. of Phys. Math. Sciences Yaglom for a number of valuable critical remarks, and the editor, Candidate Phys. Math. Sc. A. I. Kostienko, for careful preparation of manuscripts for the press.

## Chapter One

### GENERAL CONCEPTS OF RANDOM PROCESSES

#### §1. Dynamical and Statistical Laws

Any phenomenon occurring in nature is bound up with an infinite set of other phenomena. Choosing any such phenomenon as subject of study, we find that among these connections there are essential ones, which determine the basic features of the phenomenon under study, but there are also nonessential ones, which affect only some secondary features. In studying the phenomenon it is necessary to find out and take into account all the essential connections and simultaneously to disregard nonessential details caused by subsidiary connections. Thus, not the very phenomenon in all its complexity is subject to analysis, but a simplified model of it, whose behavior coincides basically with the behavior of the subject of our study in all but minor and nonessential details.

The study of the model constructed leads to the setting up of some laws. Only the abstract model of the phenomenon follows these laws. However, if the schematization of the latter has been carried out properly, the laws also describe the basic features of the phenomenon studied. Thus the criterion of correctness of a model adopted in some theory is the agreement between theoretical results and practical, experimental data.

The model of the phenomenon, later to be analyzed, should be constructed on the basis of making explicit its connections with other phenomena. Here the subdivision of factors into essential and nonessential ones depends not only on the specific nature of the phenomenon itself, but also on the problems that the theory will have to solve.

At present a great many schemes of physical phenomena are known, in which for definite external forces, the system's behavior is fully determined by its initial state. Such are, for instance, the free fall of a body in a gravitational field; the two- and many-body problem well known in celestial mechanics; an electric circuit with constant parameters under given excitation, etc. The laws which apply to schemes of this sort are known as dynamical. These laws are characteristic of a unique specification of the consequences of a given causes.

Beside models of phenomena which lead to the setting up of dynamical laws, other well-known models lead to the formulation of laws of a different nature -- statistical laws. To clarify this concept, let us consider as an example the model employed in the kinetic theory of gases.

The kinetic theory of gases deals with such parameters of the gas as pressure, temperature, viscosity, specific heat and others. These parameters characterize the gas as a whole and are determined by the combined action of all its molecules. A gas is an assembly of a great many molecules. In collective phenomena of thermal motion of a gas, the individual features of the behavior of its separate

particles are obliterated and the parameters mentioned are mainly of a statistical nature, i.e., are obtained as a result of averaging the effects of the individual particles. Therefore, the kinetic theory of gases can be constructed only on the basis of a statistical model of a gas, a model that allows to formulate suitable statistical laws.

At the basis of the classical kinetic theory of gases lies the following model. A vessel of arbitrary shape contains a given number of gas molecules of definite mass. Each molecule is regarded as an entirely free body not acted upon by gravity and by other molecules (otherwise than by collision with them). In the interval between collisions with other molecules or with the walls of the vessel each molecule moves in a straight line. The change of direction in collisions follows the laws of collision of elastic spheres. The initial state of the molecules is statistically characterized: their root-mean-square velocity is given and it is assumed that each molecule can with equal probability be at any point in the space enclosed by the vessel and have any direction of the velocity vector.

Side by side with the described statistical model of a gas, a dynamical model of it can also be constructed. For this it is necessary to consider a vessel of a given shape instead of an arbitrary one and also to indicate the positions and velocity vectors of all molecules at a given initial time moment.

Provided that we have at our disposal unlimited computational means, we can apply the laws of mechanics to the above-mentioned dynamical model and compute the trajectory of each molecule for a time interval of arbitrary duration. However, the computational difficulties connected with the solution of such a problem are practically insurmountable. This becomes obvious if we consider that one cubic centimeter of gas at normal pressure and temperature  $0^{\circ}\text{C}$  contains approximately  $3 \cdot 10^{19}$  molecules, the number of collisions of each being of the order of  $10^9$  per second. The impossibility of carrying out the necessary computational work shows the infeasibility of the dynamical model.

Yet another, more serious deficiency of fundamental nature affects the dynamical model in this problem. Actually the behavior of an individual molecule does not make it possible to draw any conclusions about the properties of the gas as a whole, whilst these are the very properties with which the kinetic theory is concerned. Consequently, in this case the dynamical model is in principle unsuitable for establishing the laws in which we are interested.

In the case examined, the statistical approach is necessitated by the fact that a very large number of particles participate in the phenomenon. We shall show another example in which the statistical treatment is expedient, but where the collective character of the phenomenon is somewhat different. This example refers to radio-reception in the presence of random interference.

If the form of the interference acting at the time when useful signals are transmitted is known, then, knowing the construction and parameters of the radio-receiver, one can always calculate with more or less labor the distortion of the useful signals by the interference. However, the results thus obtained are of no essential value for the theory of radio reception, since they are related to the particular case at hand and do not make it possible to draw general conclusions about the effectiveness of any given radio-receiving apparatus.

In contrast, the statistical approach to the problem makes it possible to establish what distortions appear on the average (e.g., the average number of distorted telegraph signs) for a given interference level. This makes it possible to

get an idea of the quality of the radio-reception method applied. Here the collective character of the phenomenon lies in that the interference effects are investigated for recurrent reception conditions, which on the average show no variation.

The following conclusions can be drawn from the foregoing. If the course of the physical phenomenon is mainly determined by a small number of principal causes (necessary events), then the dynamical model is the one most suitable for its study. Besides necessary events, we encounter other physical phenomena whose basic features are determined by an exceedingly great number of factors which are on the average of approximately equal effect (random events). We stress that a random effect is just as causally determined as a certain event, but it differs from the latter in the character of its causes.

It is impossible to construct a dynamical model of a random phenomenon, to establish the laws which govern the individual random event, or to make an even rough prediction of its course.

The laws of random phenomena are revealed by the observation of a great number of events taking place under similar conditions or by their multiple recurrence. These laws are qualitatively different from the laws which govern the individual event. They are statistical and are studied with the aid of probability models of these events.

## §2. Random Processes

The random events studied by the classical theory of probability are events which can either occur or not occur when a certain complex of conditions is realized. Such are, for instance, the obtaining of a given number of points in casting dice, the emission of a given elementary particle in a given time interval by a radioactive atom, the distortion of an individual telegraph sign by the random interference in the communication channel, etc. The development of physics and engineering has made it necessary to study phenomena of a different type—the random events continuing in time, or, in other words, random processes.

One of the first random processes studied by physicists was the Brownian movement, i.e., the movement of minute particles suspended in a liquid or gas, discovered in the year 1827 by the English botanist Brown. At the beginning of the twentieth century, simultaneously with the elaboration of the theory of the Brownian movement, the study of random voltages and currents in electric circuits resulting from thermal agitation in their elements was begun. In the nineteen-twenties these investigations were extended to circuits with electronic tubes.

In all the examples mentioned, the random process was the result of thermal movement of matter. However, it is easy to indicate a great number of random processes caused by quite different factors. Such are, for instance, earth displacements in quakes, ships rolling on rough sea, vibration of vehicles in motion on an uneven road, acoustic noise, time variation of meteorological factors, variations in the load on a mains network supplying many consumers, etc.

The first steps in the theory of random processes, dating back to the beginning of the twentieth century, are typical of the fact that for each problem encountered a specific method of solution was evolved, suitable only for the given problem or a narrow circle of related problems. The general theory of random processes appeared much later. Its foundations were laid by the works of the Soviet mathematicians A. M. Kolmogorov and A. Ya. Khinchin published in the nineteen-thirties.

The problems dealt with by the theory of random processes can be subdivided into two major groups. The first group contains the problems connected with the mechanism by which the random processes are produced. Although some general methods of investigating the problems of this group can be indicated, the specific physical content of each problem generally shows through.

The second group contains problems relating to random influence upon systems of various physical nature. Here dynamic analogies exist which make it possible to describe phenomena widely divergent in their nature, (mechanical, electrical, acoustic phenomena) with the aid of a standard mathematical apparatus. These analogies make this field of the theory of random processes similar in character to the theory of oscillations which studies oscillatory processes in various branches of physics and engineering from a common point of view.

The present work examines problems of the second group only, i.e., the behavior of various systems under given random influences is studied. The mechanism by which these influences are produced is not examined in detail.

### §3. The Role of the Theory of Random Processes in Engineering

The installations, apparatus and devices of modern engineering are systems functioning under some external forces. This applies equally to their assemblies and elements. Examples are: a bridge subjected to the load of vehicles and pedestrians, apparatus under electric tension, the rope of lifting gear subjected to strain, and many others. In accordance with §1 the analysis of phenomena in the systems mentioned is preceded by schematization of the object of study. This schematization should be applied to the properties of the system and to the properties of the external influence [input].

The methods of constructing schematized models of real systems are outside the scope of the problems dealt with in this book. We shall, therefore, dwell only on methods of schematizing external forces.

The majority of engineering apparatus is intended for repeated use under similar working conditions. Therefore, their behavior is studied under external influences of a known character, determined by these conditions.

Similar though they might be, operating conditions are never identical. The same can be said of the external forces. The problem of schematization lies in that, that all essential properties and peculiarities of the input should be considered while disregarding all its secondary features. Depending on the character of the input, dynamic schematization is more convenient in some cases, and statistical in others.

Dynamic schematization of the input is its representation by some well-defined function of the time, whose form is established by analyzing the working conditions. Such idealization is expedient when the form of the input varies only inconsiderably from case to case. Let us give one example. The alternating voltage of the mains network, to which the device investigated is connected does not have a strictly constant amplitude, frequency and form. However, in most cases, this voltage can be successfully idealized, and be considered sinusoidal, having well-defined amplitude and frequency.

It is rational to use the statistical schematization of an input, i.e., to consider it as if it were some random process, for which the investigation of the operating conditions gives only probability characteristics, when a great variety of essentially different forms of input occurs.

Some such inputs are readily perceived in the examples of the preceding section. We shall supplement them by one additional example, where dynamic schematization of the input is possible, but the statistical one is more appropriate.

Until recently, in communication engineering, when analyzing some apparatus or its individual elements, the signals transmitted along the channel were regarded as determinate time functions of a certain form (dynamic schematization). Such an approach to the problem is only the first approximation to reality. The point is that any definite time function transmitted through the channel in a given interval constitutes but one of the possible variants of the signal. The communication apparatus should be designed for all the possible variants of a signal.

In most cases the number of possible variants of the signal is so great that any attempt at a simultaneous or successive consideration of all of them is exceedingly difficult. The situation becomes even more aggravated by the fact that noise of extremely irregular character is superimposed upon the signal in the communication channel. Hence follows naturally the conclusion that it is more expedient to regard the signal transmitted through the communication channel as a random process, i. e., to apply a statistical schematization.

The examination of the external forces on various engineering apparatus shows that in many cases the statistical schematization represents more closely their properties than does dynamic schematization. The same applies to many problems of the theory of communication and automatic control, the theory of recording devices, the theory of vibration, shocks and rolling experienced by structures, vehicles, vessels and airplanes, and to many other cases. The statistical approach to such problems is not yet widely used, mainly owing to its comparative novelty and to the related fact that the statistical properties of various random actions are not well known. This latter fact has also influenced the contents of this monograph, limiting the variety of specific technical examples analyzed in it.

## Chapter Two

### RANDOM FUNCTIONS AND THEIR LINEAR TRANSFORMATIONS

#### §4. The Concept of Random Function

The mathematical image of a physical random process is the random function. Our exposition of the theory of random processes in linear physical systems is therefore headed by the present chapter, in which we state the basic properties of random functions and their linear transformations.

The random process gives in each of a great number of experiments, carried out under similar conditions, a time function  $f_k(t)$ , where  $k$  is the number of the corresponding experiment. In the course of an individual experiment, a well-defined value of  $f_k(t)$  corresponds to each value of the argument  $t$ . Thus  $f_k(t)$  is a determinate function. It is called the realization of the random process  $f(t)$  in the  $k$ -th experiment. The random nature of the process is manifested by the fact that the form of the function  $f_k(t)$  varies at random from experiment to experiment. To characterize the random process it is necessary to indicate all its possible realizations and their probabilities.

In the mathematical theory of random processes the following more rigorous formulation of the aforesaid follows from the concepts developed by A. N. Kolmogorov /1/ in his axiomatic construction of the theory of probability: the random function  $f(t)$  is defined if a probability measure is given on the set of its realizations.

Thus, the random function is a function of two variables: the time  $t$  and the parameter  $k$ , enumerating all the possible realizations. For any particular value of  $k$  the function  $f_k(t)$  is a determinate function. In contrast, if the time  $t$  is fixed, the function becomes a random variable.

Let us note that among the possible values of  $t$  there may be such particular values for which the random function equals a constant. For instance, for the random function which represents the response of some system to a random input, such a particular value may be initial moment of this input.

We shall examine a somewhat different method of defining the random function that we shall use below. This method is due to E. E. Slutskii /2/. It can be shown that this definition is included in the former as a particular case.

The random variable  $f = f(t)$  corresponding to a selected time instant  $t$  is fully defined if its distribution function or probability density  $w(t)$  is given. To characterize the random function, the mentioned distribution function or probability density should be given for any time instant which lies within the time limits of observation of the random process.

This characterization of the random function is still incomplete. The random variables  $f_1 = f(t_1)$ ,  $f_2 = f(t_2)$ , ...,  $f_n = f(t_n)$  which correspond to the various time instants  $t_1, t_2, \dots, t_n$ , are in the general case statistically dependent. The presence of this statistical dependence makes it necessary to specify an n-dimensional distribution law  $w(f_1, \dots, f_2, \dots, f_n)$ . Here the number n has an arbitrary value. For each n the time instants  $t_1, t_2, \dots, t_n$  can be arbitrarily chosen within the limits of observation time.

Thus, the random function is defined, if for any value of n an n-dimensional distribution law  $w(f_1, f_2, \dots, f_n)$  is given, where the time instants  $t_1, t_2, \dots, t_n$  can be arbitrarily distributed within the limits of the time of observation.

In many cases this complete characterization of the random function is superfluous. The calculations necessary for practical purposes can be often carried out if only the two-dimensional probability density  $w(f_1, f_2)$  is known. It is sufficient for the solution of some problems to know only the one-dimensional probability density  $w(f)$ . This is the case, for instance, if it is required to find the probability that f shall exceed a given value a.

### §5. Stationary and Nonstationary Random Functions

Let the n-dimensional distribution of the random function be given. We shall fix the mutual positions of the corresponding n points  $t_1, t_2, \dots, t_n$  on the time axis and then displace the set of these points along the time axis without changing their relative positions. If all the distributions which determine the random function remain constant under this displacement, the random function is called stationary. Thus, a stationary random function is statistically invariant with respect to time-translation.

For a stationary random function the one-dimensional probability distribution is independent of the time t, the two dimensional distribution is dependent on the difference  $t_2 - t_1$  only, etc.

In the general case the said time-homogeneity does not hold, and the random function and random process described by it are nonstationary. The multi-dimensional probability density of a nonstationary random function depends on the position of each of the n time instants.

The random functions met with in many physical and engineering problems can be considered, with sufficient accuracy, as stationary. Stationary processes therefore occupy an important place in the modern theory of random processes. In a number of cases, however, the random process is essentially nonstationary. This hinders us from limiting the theory to the examination of only stationary processes.

### §6. Moments

As known from probability theory, the full characterization of a random variable is given by its distribution law. Sometimes, however, such exhaustive information on the random variable is superfluous. In this case the values of the first few moments of the distribution law are often indicated instead of the distribution law itself. An analogous situation exists in the theory of random processes. Here

one often refrains from considering n-dimensional distribution laws and operates with a finite number of moments.

The description of a random process with the aid of moments is less comprehensive than in using suitable distribution laws. However, in many problems of the theory of random processes probability densities are rather difficult to obtain, while moments are computed by quite elementary means and describe sufficiently well the phenomenon studied. Moments are therefore widely used in the theory of random processes.

Moments are obtained by the averaging operation. In connection with this we shall first examine the question of averaging random functions.

As was shown in §4 the value of a random function depends on time and the parameter  $k$  (realization number). One of the possible methods of computing an average is by fixing a definite time instant and computing the average of the totality of realizations of the random function. Here the operation of averaging can be applied not only to the values of the random function, but to any function of it. Exactly in the same way one can select several time instants and apply the averaging operation to any function of the corresponding values of the random function.

Another possibility is: selecting a certain number of realizations of the random function and calculating their time-averages. As in the preceding case the averaging operation can be applied to the value of the random function, to any function of this value or of several such values at once. In the latter case it is first necessary to fix definite positions of the corresponding time for all time-axis points, and to displace (in the course of calculation) these points along the time axis, without changing their relative positions.

Consequently, a random function has average values with respect to the set of its realizations, and time-average values. The first correspond to definite time instants, the second--to a definite realization of the random function.

In the general case, when the random process is nonstationary, the only average values of the first kind are of interest. These values are determined by the set of possible realizations of the random function and characterize the latter as a whole. For a stationary process the time-average values are also considered.

We now proceed to the definition of moments, starting with a one-dimensional distribution  $w(f)$  at a time instant  $t$ . The moment of order  $\nu$  of this distribution is defined by:

$$m_{\nu}(t) = \int_{-\infty}^{+\infty} f^{\nu} w(f) df \quad (\nu = 1, 2, 3, \dots). \quad (2.1)$$

It follows from expression (2.1) that the moment  $m_{\nu}(t)$  is the average of the random variable  $f^{\nu}(t)$  over the set of realizations of the random function. This average is called the mathematical expectation of the random function  $f^{\nu}(t)$  at time  $t$ . We shall henceforth denote the mathematical expectation of a random quantity  $A$  by the symbol  $M[A]$ . Accordingly

$$m_{\nu}(t) = M[f^{\nu}(t)]. \quad (2.2)$$

In general, the moments (2.2) are time-dependent and are constants for stationary random functions.

The simplest moment is the moment of first order or first moment

$$m_1(t) = M[f(t)] \quad (2.3)$$

which is the average of the random function at the time  $t$  over its set of realizations.

The second moment

$$m_2(t) = M[f^2(t)] \quad (2.4)$$

is the average of the square of the random function at the time  $t$  over its set of realizations.

For the solution of numerous practical problems it is often sufficient to consider only the first two moments of the random function. In some cases such a characterization of the random process is insufficient and it is necessary to have recourse to moments of a higher order, i.e., the third and fourth.

The moments of multi-dimensional distributions are introduced in an analogous way. The moment of order  $v$  of the two-dimensional distribution  $w(f_1, f_2)$  will be any function of the form

$$m_{ij}(t_1, t_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_1^i f_2^j w[f_1, f_2] df_1 df_2 = M[f_1^i \cdot f_2^j], \quad (2.5)$$

where

$$i + j = v; \quad f_1 = f(t_1); \quad f_2 = f(t_2).$$

It is easy to see that there are altogether  $v - 1$  different moments of a two-dimensional distribution function, these moments corresponding to  $i = 1, 2, 3, \dots, v - 1$ . For  $i = 0$  or  $i = v$  the moment of a two-dimensional distribution degenerates into the corresponding function of a one-dimensional distribution.

The simplest moment of a two-dimensional distribution is the mixed second order moment

$$m_{11}(t_1, t_2) = M[f(t_1) \cdot f(t_2)]. \quad (2.6)$$

The moment (2.6) is the average over the set of realizations of the product of two values of the random function corresponding to the instants  $t_1$  and  $t_2$ . If the moment  $m_{11}(t_1, t_2)$  is given, this defines the second moment (2.4) of the one-dimensional distribution, since, setting  $t_1 = t_2 = t$  in (2.6) we have

$$m_{11}(t, t) = M[f^2(t)] = m_2(t). \quad (2.7)$$

For a stationary random function of the moments (2.5) are functions of the two variables  $t_1$  and  $t_2$ . For a stationary function a time-translation leave the probabilities of the process characteristics unaltered and hence the moments are functions of only the time interval  $\tau = |t_2 - t_1|$ .

Any moment of order  $v$  of the three-dimensional distribution  $w(f_1, f_2, f_3)$  is expressed by:

$$m_{ijk}(t_1, t_2, t_3) = M[f^i(t_1) \cdot f^j(t_2) \cdot f^k(t_3)], \quad (2.8)$$

where

$$i + j + k = v.$$

We shall note the fact that stating the third order moment of the three-dimensional distribution

$$m_{111}(t_1, t_2, t_3) = M[f(t_1) \cdot f(t_2) \cdot f(t_3)] \quad (2.9)$$

defines also third-order moments of a two-dimensional distribution. The transition to the latter is effected by equating the corresponding two time instants in expression (2.9). In exactly the same manner, fourth-order moments of a two-dimensional distribution are particular cases of fourth-order moments of a four-dimensional distribution.

We shall draw the following general inference from the aforesaid. A random function is defined, i.e., an  $n$ -dimensional distribution law  $w(f_1, f_2, \dots, f_n)$ , is given for any  $n$ , if for any  $v$  the moment

$$m_{\underbrace{11 \dots 1}_v}(t_1, t_2, \dots, t_v) = M[f(t_1) \cdot f(t_2) \dots f(t_v)]. \quad (2.10)$$

is known.

By equating appropriate arguments to one another we can obtain from the moments (2.10) all other moments, whilst the totality of all moments determines uniquely the  $n$ -dimensional distribution law for any  $n$ .

Since we shall chiefly meet the following moments of the type of (2.10), we shall denote for brevity:

$$m_{\underbrace{11 \dots 1}_v}(t_1, t_2, \dots, t_v) = m_v(t_1, t_2, \dots, t_v). \quad (2.11)$$

Beside the moments examined above one often meets also central moments. A central moment of order  $v$  of a  $v$ -dimensional distribution is defined by:

$$\mu_v(t_1, t_2, \dots, t_v) = M\{[f(t_1) - m_1(t_1)] [f(t_2) - m_1(t_2)] \dots [f(t_v) - m_1(t_v)]\}. \quad (2.12)$$

It follows from (2.12) that for a random function whose first moment identically vanishes, every moment is central.

A central moment of the first order always equals zero. In fact, since the mean value of a sum equals the sum of mean values of the summands, we have,

$$\mu_1(t) = M[f(t) - m_1(t)] = M[f(t)] - m_1(t) = 0. \quad (2.13)$$

We shall examine a central moment of the second order. Carrying out calculations analogous to the preceding we obtain:

$$\begin{aligned}
\mu_2(t_1, t_2) &= M\{|f(t_1) - m_1(t_1)| |f(t_2) - m_1(t_2)|\} = \\
&= M[f(t_1) \cdot f(t_2)] - m_1(t_1) M[f(t_2)] - m_1(t_2) \cdot M[f(t_1)] + \\
&\quad + m_1(t_1) \cdot m_1(t_2) = m_2(t_1, t_2) - m_1(t_1) \cdot m_1(t_2).
\end{aligned} \tag{2.14}$$

The relationships for central moments of higher order are obtained in an analogous manner.

In the particular case when  $t_1 = t_2 = t$ , expression (2.14) gives

$$\mu_2(t) = M\{|f(t) - m_1(t)|^2\} = m_2(t) - m_1^2(t). \tag{2.15}$$

The quantity  $\mu_2(t)$  is the average over the set of realizations of the square deviation of the random function from its average value  $m_1(t)$  corresponding to a time instant  $t$ , i. e., is equal to its dispersion [variance]  $\sigma^2(t)$  at this instant.

The second order central moment  $\mu_2(t_1, t_2)$  is often called autocorrelation ratio and denoted by  $k(t_1, t_2)$ . Thus,

$$k(t_1, t_2) = \mu_2(t_1, t_2). \tag{2.16}$$

The expression

$$\rho(t_1, t_2) = \frac{k(t_1, t_2)}{\sigma(t_1) \cdot \sigma(t_2)} \tag{2.17}$$

is called the normalized autocorrelation ratio. It corresponds to the correlation coefficient widely used in probability theory.

For obtaining the above-examined moments, the operation of averaging over the set of realizations of the corresponding random function was employed. This method is general and equally applicable to stationary and nonstationary random functions. For studying random functions another method of determining moments, giving the same results, is also possible.

The statistical properties of a stationary random function do not vary with time. It follows from this time-homogeneity that the average over the set of realizations calculated for a chosen time instant gives the same result as the time-average of a single realization. In the latter case the time of observation of the random process should be sufficiently long (strictly speaking, infinite). This proposition is known as the ergodic theorem.

The ergodic theorem is valid under sufficiently general conditions which are generally fulfilled in practical problems. The proof of this theorem is given in the work of A. M. Yaglom /3/ as well as in the monograph of V. I. Bunimovich /4/.

Thus, it is possible to calculate the moment of a stationary random process when a single realization of it  $f_k(t)$ , corresponding to a sufficiently long observation time, is given. The computation of, say, the first moment is carried out thus:

$$m_1 = M[f(t)] = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f_k(t) dt. \tag{2.18}$$

For the second moment of a two-dimensional distribution we have:

$$\pi_2(\tau) = M[f(t) \cdot f(t + \tau)] = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f_k(t) f_k(t + \tau) dt. \quad (2.19)$$

The computation of moments from a single realization of the observed random process has great practical advantages.

## §7. The Derivative of a Random Function

Let us consider the derivative of an individual realization  $f_k(t)$  of a random function  $f(t)$ . Since the realization  $f_k(t)$  is a determinate function of time, an analogous statement with respect to its derivative  $f'_k(t)$  is also true. Thus, the derivative  $f'_k(t)$  is a derivative in the usual sense and all the propositions of the differential calculus of determinate functions are applicable to it.

Let us now fix some time instant  $t$  and pass from one realization of the random function to another. In this case the value of the derivative  $f'_k(t)$  will vary in a random manner with the realization number. This random variable is called the derivative of the random function  $f(t)$  at the moment  $t$ . The following definition is equivalent to the above: the derivative of the random function at the moment  $t$  is the limit\*

$$\lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \frac{df(t)}{dt} = f'(t). \quad (2.20)$$

We shall explain the relation between the random function itself and its derivative. For this we shall first take a normal distribution function, i.e., a function defined by normal  $n$ -dimensional distribution laws. We shall assume for the sake of simplicity that the mentioned function is stationary. Its two-dimensional probability density is then of the form

$$w(f_1, f_2) = \frac{1}{2\pi\sigma^2 \sqrt{1-\rho^2}} \exp \left[ -\frac{f_1^2 - 2\rho f_1 f_2 + f_2^2}{2\sigma^2(1-\rho^2)} \right], \quad (2.21)$$

where we denote for the sake of brevity

$$f_1 = f(t_1); \quad f_2 = f(t_2). \quad (2.22)$$

The parameters  $\sigma^2$  and  $\rho$  which appear in (2.21) are, respectively, the dispersion and the normalized autocorrelation ratio of the random process considered, the first one being constant with respect to time by virtue of the stationary nature of the process, and the second one depending only on the time interval  $\Delta t = t_2 - t_1$ .

We shall change the variables in the distribution law (2.21) thus

$$f_1 = f; \quad f_2 = f + \Delta f. \quad (2.23)$$

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\* The conditions for the existence of the limit (2.20) will be considered below.

Hence we shall obtain after simple transformations\*

$$w[f, \Delta f] = \frac{1}{2\pi\sigma^2 \sqrt{1-\rho^2}} \exp \left[ -\frac{2f^2(1-\rho) + 2f \cdot \Delta f(1-\rho) + \Delta f^2}{2\sigma^2(1-\rho^2)} \right]. \quad (2.24)$$

In order to obtain the one-dimensional probability density for the increment  $\Delta f$  it is necessary to compute the integral

$$w(\Delta f) = \int_{-\infty}^{+\infty} w(f, \Delta f) df. \quad (2.25)$$

The value of the following definite integral is given in handbooks (see /6/) as

$$\int_{-\infty}^{+\infty} e^{-px^2 \pm qx} dx = e^{\frac{q^2}{4p}} \sqrt{\frac{\pi}{p}}. \quad (2.26)$$

Making use of (2.26) we find:

$$w(\Delta f) = \frac{1}{\sqrt{2\pi} \sqrt{2\sigma} \sqrt{1-\rho}} \exp \left[ -\frac{\Delta f^2}{2(\sqrt{2\sigma})^2(1-\rho)} \right]. \quad (2.27)$$

Thus the increment  $\Delta f$  represents a normal variable with dispersion

$$\sigma_{\Delta f}^2 = 2\sigma^2(1-\rho). \quad (2.28)$$

Consequently, if the limit (2.20) exists, then, passing to the limit for

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\* As known from probability theory /5/, the rule of the change of variables as applied to two-dimensional distribution laws is formulated in the following way. If the new random variables  $\xi$  and  $\eta$  are related to the old variables  $x$  and  $y$  by

$$\xi = f(x, y), \quad \eta = \varphi(x, y),$$

then the two-dimensional probability density  $w(\xi, \eta)$  is expressed by the two-dimensional probability density  $w(x, y)$  in the following fashion

$$w(\xi, \eta) = \frac{w(x, y)}{\frac{\partial(\xi, \eta)}{\partial(x, y)}},$$

where

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix}$$

$\Delta t \rightarrow 0$ , we shall obtain again a normal variable. Thus, the derivative of a normally distributed random function has also normal distribution.

Let us find the dispersion of the derivative. The dispersion of the quotient  $\Delta f / \Delta t$  equals

$$\sigma_{\Delta f / \Delta t}^2 = \frac{\sigma_{\Delta f}^2}{\Delta t^2} = \frac{2\sigma^2 [1 - \rho(\Delta t)]}{\Delta t^2}. \quad (2.29)$$

For  $\Delta t \rightarrow 0$  expression (2.29) becomes indeterminate since  $\rho(\Delta t) \rightarrow 1$ . To remove this indeterminacy we shall expand  $\rho(\Delta t)$  in a power series in terms of  $\Delta t$ :

$$\rho(\Delta t) = 1 + \rho'(0) \cdot \Delta t + \frac{1}{2} \rho''(0) \cdot \Delta t^2 + \dots \quad (2.30)$$

Expression (2.29) assumes now the form:

$$\sigma_{\Delta f / \Delta t}^2 = -2\sigma^2 \frac{\rho'(0) \Delta t + \frac{1}{2} \rho''(0) \Delta t^2 + \dots}{\Delta t^2}. \quad (2.31)$$

We shall examine this result. By virtue of the fact that the autocorrelation ratio  $\rho(\Delta t)$  is an even function, two cases are possible: 1) the point  $\Delta t = 0$  is an extremum, i.e.,  $\rho'(0) = 0$ ; 2) the derivative  $\rho'(0)$  at the point  $\Delta t = 0$  does not exist (such an example is the function  $\rho(\Delta t) = e^{-\alpha|\Delta t|}$ ). It is obvious that the derivative of a random function exists only in the first case. For  $\rho'(0) = 0$ , going over to the limit in (2.31) ( $\Delta t \rightarrow 0$ ) we have:

$$M \left[ \left( \frac{df}{dt} \right)^2 \right] = -\sigma^2 \rho''(0). \quad (2.32)$$

Expression (2.32), if meaningful, is always positive. Consequently, for  $\rho'(0) = 0$ , the function  $\rho(\Delta t)$  has an extremum at the point  $\Delta t = 0$ , this extremum being a maximum, since  $\rho(0) = 1$  and for  $\Delta t \neq 0$  we have  $\rho(\Delta t) < 1$ , whence it follows that we always have  $\rho''(0) < 0$ .

We shall now sum up the results of the foregoing analysis. As mentioned above, the derivative of a normally distributed random function, if it exists, has normal distribution. This result is directly related to the statement of probability theory that any linear function of several normal random variables has also a normal distribution. As the structure of (2.20) shows, the process of calculating the derivative is equivalent to forming a linear function of the two random variables  $f(t + \Delta t)$  and  $f(t)$  with subsequent passage to the limit.

It is necessary for the existence of the derivative that there should be a statistical dependence between values of the random function which are sufficiently near in time. But if values arbitrarily near in time are fully independent statistically, the probability that any realization of the random function is discontinuous at all points of the time axis and does not have a derivative is unity, i.e., the random function does not have a derivative. However, the existence of a statistical dependence is not sufficient. The statistical dependence should be such that the condition  $\rho'(0) = 0$  is fulfilled.

If we restrict ourselves to the examination of random variables of finite average values and dispersion, which case is of fundamental interest in practice, we can say that the sum of several variables having the same distribution law also obeys this law only in case the latter have normal distribution. No other distribution law has this property [7]. Hence it follows that under the condi-

tions stated differentiation does not affect the distribution law only for random functions of normal distribution.

The above reasoning is readily generalized for derivatives of higher order.

Let us find the law by which moments are transformed in differentiation. We shall first consider the first-order moment of the derivative. Since in all cases the operations of forming the average and of passage to the limit are interchangeable, we can write:

$$\begin{aligned} M\left[\frac{df}{dt}\right] &= M\left[\lim_{\Delta t \rightarrow 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}\right] = \\ &= \lim_{\Delta t \rightarrow 0} \frac{M[f(t+\Delta t)] - M[f(t)]}{\Delta t} = \frac{dm_1(t)}{dt}. \end{aligned} \quad (2.33)$$

Consequently, the first-order moment also undergoes differentiation when the random function is differentiated. It is obvious that the first-order moment of the derivative of a stationary random function equals zero.

Generalizing the result obtained for the case of the first derivative, we have:

$$M\left[\frac{d^n f}{dt^n}\right] = \frac{d^n m_1(t)}{dt^n}. \quad (2.34)$$

Let us proceed to the second-order moment of a two-dimensional distribution of the derivative. Since passing to the limit and averaging are interchangeable we can write

$$\begin{aligned} M\left[\frac{df(t_1)}{dt_1} \cdot \frac{df(t_2)}{dt_2}\right] &= \\ &= \lim_{\substack{\Delta t_1 \rightarrow 0 \\ \Delta t_2 \rightarrow 0}} M\left[\frac{f(t_1 + \Delta t_1) - f(t_1)}{\Delta t_1} \cdot \frac{f(t_2 + \Delta t_2) - f(t_2)}{\Delta t_2}\right]. \end{aligned} \quad (2.35)$$

Introducing the moment  $m_2(t_1, t_2)$  we shall give expression (2.35) the form

$$\begin{aligned} M\left[\frac{df(t_1)}{dt_1} \cdot \frac{df(t_2)}{dt_2}\right] &= \\ &= \lim_{\substack{\Delta t_1 \rightarrow 0 \\ \Delta t_2 \rightarrow 0}} \frac{1}{\Delta t_2} \left\{ \frac{m_2(t_1 + \Delta t_1, t_2 + \Delta t_2) - m_2(t_1, t_2 + \Delta t_2)}{\Delta t_1} - \right. \\ &\quad \left. - \frac{m_2(t_1 + \Delta t_1, t_2) - m_2(t_1, t_2)}{\Delta t_1} \right\}. \end{aligned} \quad (2.36)$$

The following final result is obtained from (2.36):

$$M\left[\frac{df(t_1)}{dt_1} \cdot \frac{df(t_2)}{dt_2}\right] = \frac{\partial^2 m_2(t_1, t_2)}{\partial t_1 \partial t_2}. \quad (2.37)$$

Thus, the second-order moment of a random function is differentiated twice when the function is differentiated once. Formula (2.37) is readily generalized for the case of a moment order  $v$ :

$$M\left[\frac{df(t_1)}{dt_1} \cdot \frac{df(t_2)}{dt_2} \dots \frac{df(t_v)}{dt_v}\right] = \frac{\partial^v m_v(t_1, t_2, \dots, t_v)}{\partial t_1 \partial t_2 \dots \partial t_v}. \quad (2.38)$$

In precisely the same way it is possible to obtain analogous results for higher order derivatives. The moment of order  $v$  of the  $n$ -th derivative of a random function is given by:

$$M \left[ \frac{d^n f(t_1)}{dt_1^n} \cdot \frac{d^n f(t_2)}{dt_2^n} \dots \frac{d^n f(t_s)}{dt_s^n} \right] = \frac{\partial^n m_s(t_1, t_2, \dots, t_s)}{\partial t_1^n \partial t_2^n \dots \partial t_s^n}. \quad (2.39)$$

For a stationary random functions differentiation with respect to time can be replaced by differentiation with respect to the length of interval separating time instants. As an example we shall consider the second-order moment of a two-dimensional distribution of the derivative. The right-hand side of expression (2.37) can be thus represented for a stationary function

$$\frac{\partial^2 m_2(t_1, t_2)}{\partial t_1 \partial t_2} = \frac{\partial \tau}{\partial t_2} \cdot \frac{\partial}{\partial \tau} \left\{ \frac{\partial m_2(\tau)}{\partial \tau} \cdot \frac{\partial \tau}{\partial t_1} \right\}, \quad (2.40)$$

where

$$\tau = |t_2 - t_1|.$$

Keeping in mind that the derivatives  $\partial \tau / \partial t_1$  and  $\partial \tau / \partial t_2$  equal unity in their absolute value and are of opposite sign, we obtain for a stationary random function:

$$M \left[ \frac{\partial f(t_1)}{\partial t_1} \cdot \frac{\partial f(t_2)}{\partial t_2} \right] = - \frac{d^2 m_2(\tau)}{d\tau^2}. \quad (2.41)$$

If we assume in (2.41)  $\tau = 0$ , then as could be expected we shall obtain a result coinciding with (2.32).

## §8. Integral Transformations of Random Functions

An integral transform of a random function  $f(t)$  is defined as

$$F = \int_a^b f(t) \varphi(t) dt, \quad (2.42)$$

where  $\varphi(t)$  is a determinate function.

We shall first explain the nature of the quantity  $F$ . We shall assume that a determinate function  $\varphi(t)$  is given and the integration interval  $(a, b)$  is fixed. Then operation (2.42) gives a certain numerical value of  $F$  for every realization of the random function  $f(t)$ . This value varies in a random manner from one realization to another i.e., for given  $\varphi(t)$  and integration limits the integral transform (2.42) is a random function of the realization number of the random function  $f(t)$ . The probability function of the random variable  $F$  depends on the chosen function  $\varphi(t)$  and the integration limits  $a$  and  $b$ .

For investigating the statistical properties of the random variable  $F$  we shall break up the integration interval  $(a, b)$  into elements  $\Delta t$  and go over from the integral (2.42) to the sum

$$F = \sum_{i=1}^N f(i \Delta t) \varphi(i \Delta t) \Delta t. \quad (2.43)$$

As mentioned in the preceding section, the distribution law of the sum does not, in general, coincide with the distribution law of the summands. Therefore, in general, the distribution law of the random function is altered by an integral transformation. Without examining this problem in general we shall mention two important particular cases of it.

The quantity  $F$  is normally distributed if the random function  $f(t)$  which

undergoes the integral transformation has normal distribution. The distribution of  $F$  is normal, for an arbitrary distribution law of the function  $f(t)$ , any values arbitrarily near in time are mutually independent statistically. The last result is the consequence of the central limit theorem of the theory of probability.

Let us proceed to the elucidation of the laws of transformation of moments in integral transformations of random functions. We shall begin with the first order moment  $m_1^{(F)}$  of the integral transform. Since the average of the sum equals the sum of the averages of summands, we have from (2.43):

$$m_1^{(F)} = \sum_{i=1}^N m_1^{(f)}(i \Delta t) \varphi(i \Delta t) \Delta t \quad (2.44)$$

or, passing to integral notation we obtain

$$m_1^{(F)} = \int_a^b m_1^{(f)}(t) \varphi(t) dt. \quad (2.45)$$

Thus, in integral transformations of a random function, the first-order moment undergoes an analogous integral transformation. In the particular case of a stationary random function its first-order moment does not depend upon time and expression (2.45) becomes simplified thus:

$$m_1^{(F)} = m_1^{(f)} \int_a^b \varphi(t) dt. \quad (2.46)$$

Let us calculate the second-order moment of the integral transformation (2.42). We have from (2.43):

$$\begin{aligned} F^2 &= \left[ \sum_{i=1}^N f(i \Delta t) \varphi(i \Delta t) \Delta t \right]^2 = \\ &= \sum_{i=1}^N \sum_{j=1}^N f(i \Delta t) f(j \Delta t) \varphi(i \Delta t) \varphi(j \Delta t) (\Delta t)^2. \end{aligned} \quad (2.47)$$

With respect to an individual term of the sum (2.47), the operation of averaging gives:

$$\begin{aligned} M[f(i \Delta t) f(j \Delta t) \varphi(i \Delta t) \varphi(j \Delta t) (\Delta t)^2] &= \\ &= M[f(i \Delta t) f(j \Delta t)] \varphi(i \Delta t) \varphi(j \Delta t) (\Delta t)^2 = \\ &= m_2^{(f)}(i \Delta t, j \Delta t) \varphi(i \Delta t) \varphi(j \Delta t) (\Delta t)^2. \end{aligned} \quad (2.48)$$

In view of this result we can write:

$$M[F^2] = m_2^{(F)} = \sum_{i=1}^N \sum_{j=1}^N m_2^{(f)}(i \Delta t, j \Delta t) \varphi(i \Delta t) \varphi(j \Delta t) (\Delta t)^2 \quad (2.49)$$

or going over to the corresponding integral notation we obtain:

$$m_2^{(F)} = \int_a^b \int_a^b m_2^{(f)}(t_1, t_2) \varphi(t_1) \varphi(t_2) dt_1 dt_2. \quad (2.50)$$

Comparing this expression with (2.42) we notice that in the integral transformation (2.42), the second-order moment of a two-dimensional distribution of the random function  $f(t)$  undergoes an analogous double integral transformation.

One can readily convince oneself that it is necessary to apply a double integral transformation to the correlation function for calculating the dispersion of the integral transform F, i.e.,

$$\sigma_F^2 = \int_a^b \int_a^b k^{(f)}(t_1, t_2) \varphi(t_1) \varphi(t_2) dt_1 dt_2. \quad (2.51)$$

The generalization of the results (2.45) and (2.50) obtained above gives for the moment of order  $\nu$  of the transform F the following expression:

$$m_\nu^{(F)} = \int_a^b \dots \int_a^b m_\nu^{(f)}(t_1, t_2, \dots, t_\nu) \varphi(t_1) \varphi(t_2) \dots \varphi(t_\nu) dt_1 dt_2 \dots dt_\nu. \quad (2.52)$$

Thus the calculation of the moments of order  $\nu$  of the random variable F is carried out by means of a  $\nu$ -tuple integral transformation of the  $\nu$ -th order moment of the  $\nu$ -dimensional distribution of the random function  $f(t)$ .

#### §9. A Set of Several Integral Transformations of the Random Function

We shall examine a set of  $m$  different integral transformations of the same random function  $f(t)$ :

$$\left. \begin{aligned} F_1 &= \int_a^b f(t) \varphi_1(t) dt, \\ F_2 &= \int_a^b f(t) \varphi_2(t) dt, \\ &\dots \dots \dots \\ F_m &= \int_a^b f(t) \varphi_m(t) dt, \end{aligned} \right\} \quad (2.53)$$

where  $\varphi_1, \varphi_2, \dots, \varphi_m$  are some determinate functions.

The moments of each of the integral transforms (2.53) can be calculated with the aid of the relationships of the preceding section. The random quantities  $F_1, F_2, \dots, F_m$  are in general statistically dependent and the  $m$ -order moment of the  $m$ -dimensional distribution of these quantities is to be calculated.

Representing each of the integrals (2.53) in the form of a sum of type (2.43) we obtain:

$$F_1 F_2 \dots F_m = \left\{ \sum_{i_1=1}^N f(t_1 \Delta t) \varphi_1(t_1 \Delta t) \Delta t \right\} \times \\ \times \left\{ \sum_{i_2=1}^N f(t_2 \Delta t) \varphi_2(t_2 \Delta t) \Delta t \right\} \dots \left\{ \sum_{i_m=1}^N f(t_m \Delta t) \varphi_m(t_m \Delta t) \Delta t \right\}, \quad (2.54)$$

or, in a different manner

$$F_1 F_2 \dots F_m = \sum_{i_1=1}^N \sum_{i_2=1}^N \dots \sum_{i_m=1}^N f(t_1 \Delta t) f(t_2 \Delta t) \dots f(t_m \Delta t) \times \\ \times \varphi_1(t_1 \Delta t) \varphi_2(t_2 \Delta t) \dots \varphi_m(t_m \Delta t) (\Delta t)^m. \quad (2.55)$$

Now, carrying out the operation of averaging analogous to that of (2.48) and reverting to integral notation, we have:

$$\begin{aligned} M[F_1 F_2 \dots F_m] &= m_m^{(F)} = \\ &= \int_a^b \dots \int_a^b m_m^{(f)}(t_1, t_2, \dots, t_m) \varphi_1(t_1) \varphi_2(t_2) \dots \\ &\quad \dots \varphi_m(t_m) dt_1 dt_2 \dots dt_m. \end{aligned} \quad (2.56)$$

The result obtained can be readily generalized for different integration limits of (2.53). If these limits are correspondingly  $(a_1, b_1), (a_2, b_2), \dots, (a_m, b_m)$ , one can always choose such an interval  $(a, b)$  which contains the integration intervals of all the integrals. Each of the integrals of (2.53) can then be written as follows:

$$F_k = \int_{a_k}^{b_k} f(t) \varphi_k(t) dt = \int_a^b f(t) \varphi_k^*(t) dt, \quad (2.57)$$

where for  $a_k \leq t \leq b_k$  we have

$$\varphi_k^*(t) = \varphi_k(t), \quad (2.58)$$

and for  $a \leq t < a_k$  or  $b_k < t \leq b$  we assume

$$\varphi_k^*(t) = 0. \quad (2.59)$$

Now, writing the required result in the form of (2.56) we notice that in each integration we can return to the original limits. Thus,

$$\begin{aligned} M[F_1 F_2 \dots F_m] &= m_m^{(F)} = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \dots \int_{a_m}^{b_m} m_m^{(f)}(t_1, t_2, \dots, t_m) \times \\ &\quad \times \varphi_1(t_1) \varphi_2(t_2) \dots \varphi_m(t_m) dt_1 dt_2 \dots dt_m. \end{aligned} \quad (2.60)$$

For any pair of transformations of (2.53) we can write by virtue of (2.60):

$$m_2^{(F)} = M[F_i F_j] = \int_{a_i}^{b_i} \int_{a_j}^{b_j} m_2^{(f)}(t_i, t_j) \varphi_i(t_i) \varphi_j(t_j) dt_i dt_j. \quad (2.61)$$

Consequently, to obtain the mixed moment of a set of integral transforms it is necessary to apply successively all the transformations, defined for the random function itself, to its corresponding moment.

The result obtained allows for an obvious generalization for the case when each of the integral transformations of (2.53) refers to a specific random function and all the  $m$  random functions are statistically interdependent.

#### §10. Linear Transformations of Random Functions

The present section is concerned with the generalization of relationships obtained in preceding sections to the case of arbitrary linear transformations of random functions. These results are given in the works of V.S. Pugachev [8, 9].

For the sake of clear and brief notation of the forthcoming theorems we shall use operational formalism. We shall denote an operation performed upon a variable or function  $x$  as the product of this variable or function and the corresponding operator, e.g., the operator  $A$ . Thus, the result of carrying out the operation  $A$  upon a variable or function  $x$  is symbolically expressed as follows:

$$y = Ax. \quad (2.62)$$

The operation  $A$  can vary greatly from case to case. It can, for instance, determine a function  $y(x)$ , i.e., indicate a rule according to which the value of the variable  $y$  is found from the value of  $x$ . The integral transformations examined in the preceding sections bear a different character, these transformations being operators which determine the quantity  $F$  from the given function  $f(t)$ . The transition from a function  $f(t)$  to another function  $\varphi(t)$  carried out in a certain way can also constitute such an operation. An example of such an operation is differentiation. Finally, it is not difficult to think of a great number of various operations in which a function  $y(t)$  is determined by the value of a variable  $x$ .

From all these possible operations we shall pick out linear operations, i.e., operations which have the property

$$\begin{aligned} A(a_1x_1 + a_2x_2 + \dots + a_nx_n) &= \\ &= a_1Ax_1 + a_2Ax_2 + \dots + a_nAx_n. \end{aligned} \quad (2.63)$$

Equation (2.63) signifies that the operation  $A$  upon a linear function of the variables or functions  $x_1, x_2, \dots, x_n$  is a linear function of the results of this operation upon each of these variables or functions. An operator which fulfils relationship (2.63) is called linear.

We shall call an arbitrary linear operation carried out upon a random function a linear transformation of the random function. It is easy to see that differentiation and integral transformations of the random function are particular cases of its linear transformation.

We shall examine an arbitrary linear transformation of the random function  $f(t)$ :

$$F(z) = A \cdot f(t). \quad (2.64)$$

The argument may remain invariant ( $z = t$ ) in the linear transformation (2.64) as, e.g., in differentiation, and may change as was the case with the integral transformation.

If the function undergoing a linear transformation has normal distribution, the outcome of the transformation is again a normally distributed variable or function. For any other distribution law having finite mean and dispersion, the distribution law is changed by the linear transformation.

As to the operations of differentiation and integral transformation, we have noted that the  $n$ -th moment of the image of  $f(t)$  under a linear transformation  $A$  equals the  $n$ -fold iteration of  $A$  applied to the  $n$ -th moment of  $f(t)$ :

$$m_n^{(F)}(z_1, z_2, \dots, z_n) = A_{t_1} A_{t_2} \dots A_{t_n} m_n^{(f)}(t_1, t_2, \dots, t_n). \quad (2.65)$$

In the proof of relationship (2.65) with respect to both these operations the fact was used that the order in which the operation of differentiation (or integral transformation) and the operation of averaging are carried out is immaterial (the commutative property of these operations). Since the operation of averaging commutes with any linear operation, (2.65) applies to all linear transformations of a random function.

As an example we shall apply result (2.65) to a linear differential equation of  $n$ -th order

$$A_t y = B_t x. \quad (2.66)$$

where  $x = x(t)$  is a known random function,  $y = y(t)$  is an unknown random function,  $A_t$  and  $B_t$  are linear differential operators which written out equal:

$$A_t = a_n \frac{d^n}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \dots + a_1 \frac{d}{dt} + a_0, \quad (2.67)$$

$$B_t = b_m \frac{d^m}{dt^m} + b_{m-1} \frac{d^{m-1}}{dt^{m-1}} + \dots + b_1 \frac{d}{dt} + b_0. \quad (2.68)$$

Equation (2.66) in which the functions  $x(t)$  and  $y(t)$  are random is called a stochastic differential equation.

In accordance with the results given above, a differential equation analogous to (2.66) is valid for the first-order moments  $m_1^{(x)}(t)$  and  $m_1^{(y)}(t)$ :

$$A_t m_1^{(y)}(t) = B_t m_1^{(x)}(t). \quad (2.69)$$

For mixed second-order moments we have:

$$A_{t_1} A_{t_2} m_2^{(y)}(t_1, t_2) = B_{t_1} B_{t_2} m_2^{(x)}(t_1, t_2), \quad (2.70)$$

or, in full:

$$\begin{aligned} & \left( a_n \frac{\partial^n}{\partial t_1^n} + \dots + a_0 \right) \left( a_n \frac{\partial^n}{\partial t_2^n} + \dots + a_0 \right) m_2^{(y)}(t_1, t_2) = \\ & = \left( b_m \frac{\partial^m}{\partial t_1^m} + \dots + b_0 \right) \left( b_m \frac{\partial^m}{\partial t_2^m} + \dots + b_0 \right) m_2^{(x)}(t_1, t_2). \end{aligned} \quad (2.71)$$

The coefficients  $a_1$  and  $b_1$  in operators (2.67) and (2.68) and, consequently, in equations (2.69), (2.70) can be either constants or determinate time functions.

Thus for the moments of the first and second order we have obtained: in the first case an ordinary differential equation, and in the second case a partial differential equation. It is easy to set up in an analogous way differential equations for moments of any order.

The results given in §§7 and 8 have been generalized above for the case of arbitrary linear transformations. The results given in §9 are also capable of an analogous generalization.

Let a set of arbitrary linear transformations of a random function be given:

$$\left. \begin{aligned} F_1(z_1) &= A_1 f(t), \\ F_2(z_2) &= A_2 f(t), \\ &\dots \dots \dots \\ F_m(z_m) &= A_m f(t). \end{aligned} \right\} \quad (2.72)$$

Then we have by analogy with equation (2.60)

$$\begin{aligned} M[F_1 F_2 \dots F_m] &= m_m^{(F)} = A_{1t_1} A_{2t_2} \dots \\ &\dots A_{mt_m} m_m^{(f)}(t_1, t_2, \dots, t_m). \end{aligned} \quad (2.73)$$

When all the transformations of (2.72) contain different random functions which are statistically interdependent the result is written in an analogous way.

## Chapter Three

### RANDOM FORCES UPON A LINEAR SYSTEM WITH LUMPED CONSTANTS

#### § 11. Statement of the Problem and Terminology

The processes which occur in electrical, mechanical, and electromechanical systems are described by similar differential equations. For any system of the mentioned three types an equivalent system of any one of the two other types can be given. This makes it possible to construct general methods of analysis of random processes, equally applicable to any of the indicated classes. Consequently, we shall henceforth not go into the physical method of realizing a system subject to random action but shall examine the problem in its most general formulation.

The locus of points where random force is applied to the system is called its input. Depending upon the nature of the random process studied, the external force can be of a different physical nature: an electrical voltage or current, a mechanical force, torque etc. Similarly, the concept of the "locus in the system" should be made more exact in specific cases according to the method of realization of the system. Such are: a pair of terminals of an electrical system across which some electric voltage is applied; a point executing translatory motion in a mechanical system, at which point the mechanical force is applied; the axis of rotational motion in a mechanical system when a torque is exerted on this axis.

The locus of points at which the response to the applied force is observed, is called the output of the system. The response of the system may be, for instance, a voltage or current in an electrical system, a displacement or angle of rotation in a mechanical system.

The simplest problem is the investigation of a random process in a system with one input and one output. It is assumed in this case that a sufficiently complete characterization of the input applied to the system is available. The statistical properties of the system's response are here the subject of study. It is also important in some cases to study the statistical correlation between response and external force. The question, what kind of statistical characteristic of the external force is sufficient, is solved with regard to the required exactitude of the statistical properties of the system's response.

The general case is that of a system of  $m$  inputs and  $n$  outputs. Of interest here are not only the statistical properties of each of the responses separately but also the nature of the statistical correlation between the responses at different outputs as well as that between responses and external forces.

The course of the random process in the system depends upon its state at the initial time when input is applied, i. e., upon initial conditions. The latter may be specified uniquely, as is done in the theory of dynamic processes, or statistically. Only in a particular (but important) case, where only the asymptotical behavior of the system for  $t \rightarrow \infty$  is studied, is the knowledge of initial conditions superfluous.

## § 12. The Method of Stochastic Differential Equations

Let us consider a random process in a system having one input and one output. As well known, in a dynamic process the response  $y(t)$  of the linear system with constant lumped parameters under external force  $x(t)$ , satisfies an ordinary differential equation with constant coefficients:

$$A_t y(t) = B_t x(t). \quad (3.1)$$

Here  $A_t$  and  $B_t$  are linear differential operators:

$$A_t = a_n \frac{d^n}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \dots + a_1 \frac{d}{dt} + a_0, \quad (3.2)$$

$$B_t = b_m \frac{d^m}{dt^m} + b_{m-1} \frac{d^{m-1}}{dt^{m-1}} + \dots + b_1 \frac{d}{dt} + b_0. \quad (3.3)$$

Equation (3.1) is equally applicable to the description of a random process in the system. Then the external force  $x(t)$  is a known random function of time and the problem consists in investigating of the statistical properties of the response  $y(t)$  at the output of the system. Equation (3.1) then becomes a stochastic equation.

As had been already shown in § 10, a transformation of the distribution law takes place generally in a linear system. Without discussing the problem of the transformation of distribution laws in linear systems we shall turn now to the much simpler problem of the transformation of the moments induced thereby.

Using the results of § 10 which deals with linear stochastic differential equations, one can set up differential equations of the usual kind, corresponding to the stochastic equation (3.1) and which describe the behavior of the moments of the response  $y(t)$ .

For first-order moments we have the usual ordinary differential equation of  $n$ -th order:

$$A_t m_1^{(y)}(t) = B_t m_1^{(x)}(t). \quad (3.4)$$

Let us examine some particular cases of equation (3.4). The simplest case is the one in which the input is stationary, i. e.,  $m_1^{(x)} = \text{const.}$  and the first moment of the steady-state response, i. e., at  $t \rightarrow \infty$ , is required. Then the operators  $A_t$  and  $B_t$  are simplified in the following manner:

$$A_t = a_0, \quad B_t = b_0, \quad (3.5)$$

and the required first-order moment is:

$$m_1^{(y)} = \frac{b_0}{a_0} m_1^{(x)}. \quad (3.6)$$

If the input is stationary but the process leading to the attainment of a steady state is of interest, one has to superpose upon the steady-state solution for the first-order moment given by (3.6) a transient process described by the homogeneous differential equation:

$$A_t m_1^{(y)}(t) = 0. \quad (3.7)$$

The solution here obtained of equation (3.4) with constant right-hand side should satisfy the initial conditions of the system.

If the first-order moment of the input varies in time in a known way and it is required to obtain a full description of the behavior of the first-order moment of the response, the most general form of equation (3.4) should be used. The time variation of the first-order moment of the input may be the result of either the nonstationary character of the random force itself, or of the superposition of a regular input upon the random process. A mixed case, when both of the mentioned factors appear, is also possible.

The solution of equation (3.4) can be effected by any of the known methods. Since the random nature of the input does not introduce any new features into the solution, we shall not dwell upon its technical aspect.

Let us turn to second-order moments. The second-order moments of the input  $m_2^{(x)}(t_1, t_2)$  is, in general, a function of two variables, as is the second-order moment of the response of the system. It was shown in § 10 that the mentioned moments are connected by the differential equation (2.70) whose explicit form is given by the expression (2.71). Thus:

$$A_{t_1} A_{t_2} m_2^{(y)}(t_1, t_2) = B_{t_1} B_{t_2} m_2^{(x)}(t_1, t_2), \quad (3.8)$$

where  $A_{t_1}$ ,  $B_{t_1}$ , and  $A_{t_2}$ ,  $B_{t_2}$  are linear differential operators of the form of (3.2) and (3.3), where the variable  $t$  is replaced by  $t_1$  and  $t_2$  respectively.

Equation (3.8) can be solved in different ways, one of which is the following. We introduce a new unknown function of the variables  $t_1$  and  $t_2$

$$s(t_1, t_2) = A_{t_1} m_2^{(y)}(t_1, t_2) \quad (3.9)$$

and knowing the second-order moment of the input, we find the function

$$z(t_1, t_2) = B_{t_1} m_2^{(x)}(t_1, t_2). \quad (3.10)$$

Equation (3.8) is then transformed into the ordinary differential equation:

$$A_{t_1} s(t_1, t_2) = B_{t_1} z(t_1, t_2). \quad (3.11)$$

In this equation the variable  $t_1$  is the argument and  $t_2$  - a parameter. The arbitrary constants appearing in its solution are functions of the parameter  $t_2$ .

After solving equation (3.11) we find the second-order moment of the system's response from the ordinary differential equation (3.9). In (3.9) the variable  $t_2$  is the argument and  $t_1$  - a parameter.

Another possible method of solving equation (3.8) is the use of the Laplace transform:

$$\bar{f}(p) = \int_0^{\infty} f(t) e^{-pt} dt. \quad (3.12)$$

Assuming, for simplicity, zero initial conditions, let us apply the Laplace transformation with respect to the variable  $t_1$  to both sides of equation (3.8). We obtain:

$$\bar{A}_p A_{t_2} \bar{m}_2^{(y)}(p, t_2) = \bar{B}_p B_{t_2} \bar{m}_2^{(x)}(p, t_2), \quad (3.13)$$

where

$$\bar{A}_p = a_n p_1^n + a_{n-1} p_1^{n-1} + \dots + a_1 p_1 + a_0, \quad (3.14)$$

$$\bar{B}_p = b_m p_1^m + b_{m-1} p_1^{m-1} + \dots + b_1 p_1 + b_0, \quad (3.15)$$

$$\bar{m}_2^{(y)}(p_1, t_2) = \int_0^\infty m_2^{(y)}(t_1, t_2) e^{-p_1 t_1} dt_1, \quad (3.16)$$

$$\bar{m}_2^{(x)}(p_1, t_2) = \int_0^\infty m_2^{(x)}(t_1, t_2) e^{-p_1 t_1} dt_1. \quad (3.17)$$

The application of a second Laplace transformation with respect to the variable  $t_2$  gives

$$\bar{A}_p \bar{A}_p \bar{m}_2^{(y)}(p_1, p_2) = \bar{B}_p \bar{B}_p \bar{m}_2^{(x)}(p_1, p_2), \quad (3.18)$$

where  $\bar{A}_{p_2}$  and  $\bar{B}_{p_2}$  are expressed by formulas (3.14) and (3.15) with  $p_1$  replaced by  $p_2$ , and the twice-transformed moments have the form

$$\begin{aligned} \bar{m}_2^{(x)}(p_1, p_2) &= \int_0^\infty \bar{m}_2^{(x)}(p_1, t_2) e^{-p_2 t_2} dt_2 = \\ &= \int_0^\infty \int_0^\infty m_2^{(x)}(t_1, t_2) e^{-(p_1 t_1 + p_2 t_2)} dt_1 dt_2, \end{aligned} \quad (3.19)$$

$$\begin{aligned} \bar{m}_2^{(y)}(p_1, p_2) &= \int_0^\infty \bar{m}_2^{(y)}(p_1, t_2) e^{-p_2 t_2} dt_2 = \\ &= \int_0^\infty \int_0^\infty m_2^{(y)}(t_1, t_2) e^{-(p_1 t_1 + p_2 t_2)} dt_1 dt_2. \end{aligned} \quad (3.20)$$

Thus we obtain from (3.18) for the "double" transform of the required moment:

$$\bar{m}_2^{(y)}(p_1, p_2) = \frac{\bar{B}_{p_2} \bar{B}_p}{\bar{A}_{p_2} \bar{A}_p} \bar{m}_2^{(x)}(p_1, p_2). \quad (3.21)$$

The transition to the function of original variables is carried out for each variable in turn and can be achieved by double application of the inverse transformation or by other methods known from operational calculus [10].

Differential equations connecting moments of higher order can be set up and solved in an analogous way.

The method of stochastic differential equations can also be applied to the more general problem of a system with  $m$  inputs and  $n$  outputs. In this case, one

has to consider a system of  $n$  stochastic differential equations instead of one such equation of the form (3.1). These equations are:

$$\sum_{i=1}^n A_i^{(ik)} y_i(t) = \sum_{j=1}^m B_i^{(jk)} x_j(t) \quad (k = 1, 2, \dots, n), \quad (3.22)$$

where  $A_i^{(ik)}$  and  $B_i^{(jk)}$  are linear differential operators,  $i$  and  $j$  are number-labels of the output and the input, respectively, and  $k$  a running index enumerating the equations of the system.

The system of conventional differential equations for the first moments

$$\sum_{i=1}^n A_i^{(ik)} m_{it}^{(y)}(t) = \sum_{j=1}^m B_i^{(jk)} m_{ij}^{(x)}(t). \quad (3.23)$$

corresponds to the system of stochastic differential equations (3.22).

By analogy with (3.8) we obtain the following system of differential equations for second-order moments:

$$\begin{aligned} \sum_{i_1=1}^n \sum_{i_2=1}^n A_{i_1}^{(k_1)} A_{i_2}^{(k_2)} m_{i_1 i_2}^{(y)}(t_1, t_2) = \\ = \sum_{j_1=1}^m \sum_{j_2=1}^m B_{i_1}^{(j_1 k_1)} B_{i_2}^{(j_2 k_2)} m_{j_1 j_2}^{(x)}(t_1, t_2) \quad (k_1, k_2 = 1, 2, \dots, n), \end{aligned} \quad (3.24)$$

where, for brevity, the notation  $m_2(t) = m(t)$  is introduced.

The moments  $m_{j_1 j_2}^{(x)}$  appearing on the right-hand side of equations (3.24) become the autocorrelation ratios at the corresponding inputs if  $j_1 = j_2 = j$ . If  $j_1 \neq j_2$  these moments are cross-correlations, i.e., they characterize the statistical dependence between the forces at different inputs. An analogous consideration applies to the moments  $m_{i_1 i_2}^{(y)}$  appearing on the left-hand side of the same equations.

### § 13. The Method of Impulse Characteristics

In a number of cases the statement of the problem of random acting on a linear system is preceded by an analysis of its response to a specific determinate input. Standard inputs are the unit step function defined by the relationships  $H(t) = 0$  for  $t \leq 0$ ,  $H(t) = 1$  for  $t > 0$ , and the unit impulse  $h(t)$  of unit area, this [delta] function being the time derivative of the unit step function. The response of the system to a unit step function is called its transfer characteristic. Response to a unit impulse still lacks a generally accepted term. We shall henceforth call this response the impulse characteristic of the system. A simple relation exists between the transfer and impulse characteristics of the same system, the second being the time derivative of the first.

We shall examine first a linear system with one input and one output. We shall assume that the first  $n$  moments of the random applied input and the system's impulse characteristics  $\xi(t)$  are given. The first  $n$  moments of the response at the output of the system are to be calculated.

Let the input  $x(t)$  be applied at the initial time  $t = 0$ . We shall assume, for simplicity, that the energy of the system is zero at this moment. If this condition is not fulfilled, a damped regular process which can be calculated by known methods is superposed upon the investigated process.

We wish to find the response to the random input at the moment  $t_1$ . The applied input can be considered as a succession of contiguous elementary impulses of infinitely short duration (Figure 1). Their properties are similar to

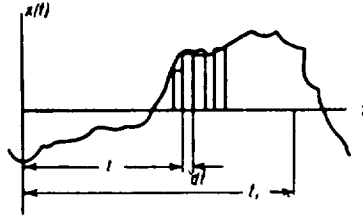


Figure 1. The applied input as a succession of contiguous impulses

those of the unit impulse. The system's response at the moment  $t_1$  to an infinitesimal impulse which is at the distance  $t$  from the origin can be expressed as follows:

$$dy(t_1) = \xi(t_1 - t) x(t) dt. \quad (3.25)$$

By virtue of the linearity of the system, the resultant response of the system to all the infinitesimal impulses  $y(t_1)$ , contained in the interval  $(0, t_1)$ , is the superposition of the corresponding elementary responses of the form (3.25). Thus:

$$y(t_1) = \int_0^{t_1} x(t) \xi(t_1 - t) dt, \quad (3.26)$$

Expression (3.26) is an integral transformation of the random function  $x(t)$ . If the first  $n$  moments of the input  $x(t)$  are known, then the general equation (2.52) of § 8 makes it possible to find the first  $n$  moments of the response  $y(t)$ . In particular, we have on the basis of (2.45) for the first moment  $y(t_1)$ :

$$m_1^{(y)}(t_1) = \int_0^{t_1} m_1^{(x)}(t) \xi(t_1 - t) dt. \quad (3.27)$$

The second moment [variance] of response  $y(t_1)$  is determined from formula (2.50) as follows

$$m_2^{(y)}(t_1) = \int_0^{t_1} \int_0^{t_1} m_2^{(x)}(t'_1, t'_2) \xi(t_1 - t'_1) \xi(t_1 - t'_2) dt'_1 dt'_2. \quad (3.28)$$

For finding the general form of the moments of the system's response we shall proceed as follows. By analogy with (3.26), the response of the system at a time instant  $t_2$  can be written in the form:

$$y(t_2) = \int_0^{t_2} x(t) \xi(t_2 - t) dt. \quad (3.29)$$

The system's response at any other time instants  $t_3, t_4, \dots, t_n$ , can be expressed just as easily. Considering all these expressions as a set of  $n$  integral transformations of the random function  $x(t)$  which are similar to (2.53), it is not difficult to calculate, using expression (2.60), the  $n$ -th order moment of the system's response. In particular, we have for the second-order moment:

$$m_2^{(y)}(t_1, t_2) = \int_0^{t_1} \int_0^{t_2} m_2^{(x)}(t'_1, t'_2) \xi(t_1 - t'_1) \xi(t_2 - t'_2) dt'_1 dt'_2. \quad (3.30)$$

Thus, being given the moments of a random input and the system's impulse characteristic, and employing the properties of integral transformations, we can calculate the corresponding moments of the system's response to the mentioned random input.

The above-listed relationships for moments make it possible to carry out calculations in the most general case when the response is nonstationary and caused by a nonstationary external force. When only the values of the moments of response at an infinitely long time after the input is switched on are of interest, the lower limits of all the integrals of this section should be taken as  $-\infty$ . Then the precise values of the upper limits become irrelevant. Only their correct ordering in time is important.

According to the remark made in § 8, the response of a system to a normally distributed applied input is also normally distributed. In the general case, an alteration of the distribution law takes place in a linear system.

The problem of random action upon a system with one input and several outputs is solved in an equally simple manner. The responses of the system  $y_1, y_2, \dots, y_n$  at its  $n$  outputs at time instants  $t_1, t_2, \dots, t_n$  are written by analogy with (3.26) and (3.29) as follows:

$$\left. \begin{aligned} y_1(t_1) &= \int_0^{t_1} x(t) \xi_1(t_1 - t) dt, \\ y_2(t_2) &= \int_0^{t_2} x(t) \xi_2(t_2 - t) dt, \\ &\dots \dots \dots \\ y_n(t_n) &= \int_0^{t_n} x(t) \xi_n(t_n - t) dt, \end{aligned} \right\} \quad (3.31)$$

where  $\xi_1(t), \xi_2(t), \dots, \xi_n(t)$  are impulse characteristics of the system with respect to the corresponding outputs.

The moments of each response are determined by relationships of type (3.27) and (3.30). The mixed moment for the system's  $n$  responses can be found if we consider the expression (3.31) as a set of integral transformations of (2.53). Then the application of formula (2.60) gives the following result:

$$\begin{aligned} m_n^{(y_1, y_2, \dots, y_n)}(t_1, t_2, \dots, t_n) &= \int_0^{t_1} \int_0^{t_2} \dots \int_0^{t_n} m_n^{(x)}(t_1, t_2, \dots, t_n) \times \\ &\times \xi_1(t_1 - t'_1) \xi_2(t_2 - t'_2) \dots \xi_n(t_n - t'_n) dt'_1 dt'_2 \dots dt'_n. \end{aligned} \quad (3.32)$$

Let us now consider a system with two inputs and two outputs. If convenient, the results of its analysis can be generalized in a quite obvious way for the case of  $m$  inputs and  $n$  outputs.

We now introduce the impulse characteristic  $\xi_{ij}(t)$  of the system, where the first subscript indicates the output and the second subscript the input concerned.

In our case the system is characterized by the four impulse characteristics  $\xi_{11}(t)$ ,  $\xi_{12}(t)$ ,  $\xi_{21}(t)$  and  $\xi_{22}(t)$ .

We shall call the system's response to random excitation applied at one of its inputs its partial response  $y_{ij}$ . Here, as above, the first and second subscripts refer to the output and the input, respectively. The four partial responses of a system with two inputs and outputs are written as follows:

$$\left. \begin{aligned} y_{11}(t_1) &= \int_0^{t_1} x_1(t) \xi_{11}(t_1 - t) dt, \\ y_{12}(t_1) &= \int_0^{t_1} x_2(t) \xi_{12}(t_1 - t) dt, \\ y_{21}(t_2) &= \int_0^{t_2} x_1(t) \xi_{21}(t_2 - t) dt, \\ y_{22}(t_2) &= \int_0^{t_2} x_2(t) \xi_{22}(t_2 - t) dt. \end{aligned} \right\} \quad (3.33)$$

The linearity of the system makes it possible to apply the principle of superposition and to express the total responses of the system at both outputs thus:

$$y_1(t_1) = y_{11}(t_1) + y_{12}(t_1), \quad (3.34)$$

$$y_2(t_2) = y_{21}(t_2) + y_{22}(t_2). \quad (3.35)$$

Let us examine the total response of the system at the first output. The first moment of response  $y_1(t_1)$  equals the sum of first moments of the partial responses  $y_{11}(t_1)$  and  $y_{12}(t_1)$ . The latter are determined by equations of type (3.27). Hence,

$$m_1^{(y)}(t_1) = \int_0^{t_1} m_1^{(x)}(t) \xi_{11}(t_1 - t) dt + \int_0^{t_1} m_1^{(x_2)}(t) \xi_{12}(t_1 - t) dt. \quad (3.36)$$

For finding the second moment we shall examine the response  $y_1$  at both the instants  $t_2$  and  $t_1$ :

$$y_1(t_2) = y_{11}(t_2) + y_{12}(t_2), \quad (3.37)$$

where the partial responses  $y_{11}(t_2)$  and  $y_{12}(t_2)$  are determined by the first two equations of (3.33) in which  $t_1$  is replaced by  $t_2$ . Taking into account equations (3.34) and (3.37), the required second moment can be written as follows:

$$\begin{aligned} m_2^{(y)}(t_1, t_2) &= M[y_1(t_1)y_1(t_2)] = M[y_{11}(t_1)y_{11}(t_2)] + \\ &+ M[y_{12}(t_1)y_{11}(t_2)] + M[y_{11}(t_1)y_{12}(t_2)] + M[y_{12}(t_1)y_{12}(t_2)]. \end{aligned} \quad (3.38)$$

By generalizing the result (2.60) to the case where each integral transformation (2.53) refers to a different function, we can express (3.38) in the following final form:

$$\begin{aligned}
m_2^{(y)}(t_1, t_2) = & \int_0^{t_1} \int_0^{t_2} m_2^{(x)}(t'_1, t'_2) \xi_{11}(t_1 - t'_1) \xi_{11}(t_2 - t'_2) dt'_1 dt'_2 + \\
& + \int_0^{t_1} \int_0^{t_2} m_2^{(x, x)}(t'_1, t'_2) \xi_{11}(t_2 - t'_2) \xi_{12}(t_1 - t'_1) dt'_1 dt'_2 + \\
& + \int_0^{t_1} \int_0^{t_2} m_2^{(x, x)}(t'_1, t'_2) \xi_{11}(t_1 - t'_1) \xi_{12}(t_2 - t'_2) dt'_1 dt'_2 + \\
& + \int_0^{t_1} \int_0^{t_2} m_2^{(x_2)}(t'_1, t'_2) \xi_{12}(t_1 - t'_1) \xi_{12}(t_2 - t'_2) dt'_1 dt'_2,
\end{aligned} \quad (3.39)$$

where  $m_2^{(x_1, x_2)}(t_1, t_2)$  is the mixed moment of the applied inputs  $x_1(t)$  and  $x_2(t)$ .

The calculation of moments of response at the second output is carried out in the same way. Analogously, in view of equations (3.34) and (3.35), we have for the mixed moment of responses  $y_1(t_1)$  and  $y_2(t_2)$ :

$$\begin{aligned}
m_2^{(y, y)}(t_1, t_2) = & M[y_1(t_1)y_2(t_2)] = M[y_{11}(t_1)y_{21}(t_2)] + \\
& + M[y_{12}(t_1)y_{21}(t_2)] + M[y_{11}(t_1)y_{22}(t_2)] + M[y_{12}(t_1)y_{22}(t_2)],
\end{aligned} \quad (3.40)$$

or in an expanded form:

$$\begin{aligned}
m_2^{(y, y)}(t_1, t_2) = & \int_0^{t_1} \int_0^{t_2} m_2^{(x)}(t'_1, t'_2) \xi_{11}(t_1 - t'_1) \xi_{21}(t_2 - t'_2) dt'_1 dt'_2 + \\
& + \int_0^{t_1} \int_0^{t_2} m_2^{(x, x)}(t'_1, t'_2) \xi_{12}(t_1 - t'_1) \xi_{21}(t_2 - t'_2) dt'_1 dt'_2 + \\
& + \int_0^{t_1} \int_0^{t_2} m_2^{(x, x)}(t'_1, t'_2) \xi_{11}(t_1 - t'_1) \xi_{22}(t_2 - t'_2) dt'_1 dt'_2 + \\
& + \int_0^{t_1} \int_0^{t_2} m_2^{(x_2)}(t'_1, t'_2) \xi_{12}(t_1 - t'_1) \xi_{22}(t_2 - t'_2) dt'_1 dt'_2.
\end{aligned} \quad (3.41)$$

The readily carried out generalization of expressions (3.39) and (3.41) for the case of  $m$  inputs and  $n$  outputs gives the solution of the most general problem of random input applied to a linear system.

If stationary random processes at the output of the system are analyzed, the integration limits in (3.32), (3.36), (3.39), and (3.41) should be changed in accordance with the remarks following formula (3.30).

#### § 14. The Spectral Method

Unlike the preceding sections, where the obtained results applied equally to stationary and nonstationary random processes, the random functions considered in this section are stationary.

Let the stationary function  $f(t)$  be defined in the interval  $(0, T)$  of the time axis. For simplicity, its first moment is assumed equal to zero. This random function can be represented by a Fourier series in the stated interval:

$$f(t) = \sum_{k=1}^{\infty} (A_k \cos \omega_k t + B_k \sin \omega_k t), \quad (3.42)$$

where

$$\omega_k = \frac{2\pi k}{T}, \quad (3.43)$$

$$A_k = \frac{2}{T} \int_0^T f(x) \cos \omega_k x \, dx, \quad (3.44)$$

$$B_k = \frac{2}{T} \int_0^T f(x) \sin \omega_k x \, dx. \quad (3.45)$$

The amplitudes  $A_k$  and  $B_k$  of the spectral components, obtained by the integral transformations (3.44) and (3.45) of the random function  $f(x)$  are random quantities depending on the specific realization of  $f(x)$ .

Using equation (2.50) let us find the dispersion  $\sigma_{A_k}^2$  of the amplitude  $A_k$ :

$$\sigma_{A_k}^2 = \frac{4}{T^2} \int_0^T \cos \omega_k x_2 \, dx_2 \int_0^T m_2(x_2 - x_1) \cos \omega_k x_1 \, dx_1. \quad (3.46)$$

We shall transform the double integral appearing in (3.46) as follows. Let us carry out the change of variables  $z = x_2 - x_1$ , in the inner integral.

Then

$$\begin{aligned} \sigma_{A_k}^2 &= \frac{4}{T^2} \int_0^T \cos \omega_k x_2 \, dx_2 \int_{x_2-T}^{x_2} m_2(z) \cos \omega_k (x_2 - z) \, dz = \\ &= \frac{2}{T^2} \left\{ \int_0^T dx_2 \int_{x_2-T}^{x_2} m_2(z) \cos \omega_k z \, dz + \int_0^T \cos 2\omega_k x_2 \, dx_2 \int_{x_2-T}^{x_2} m_2(z) \times \right. \\ &\quad \left. \times \cos \omega_k z \, dz + \int_0^T \sin 2\omega_k x_2 \, dx_2 \int_{x_2-T}^{x_2} m_2(z) \sin \omega_k z \, dz \right\}. \quad (3.47) \end{aligned}$$

Each of the double integrals of (3.47) has to be transformed by changing the order of integration. Considering that the function  $m_2(z)$  is even we have for the first integral:

$$\begin{aligned} &\int_0^T dx_2 \int_{x_2-T}^{x_2} m_2(z) \cos \omega_k z \, dz = \\ &= \int_{-T}^0 m_2(z) \cos \omega_k z \, dz \int_0^{z+T} dx_2 + \int_0^T m_2(z) \cos \omega_k z \, dz \int_z^T dx_2 = \\ &= 2 \int_0^T (T - z) m_2(z) \cos \omega_k z \, dz. \quad (3.48) \end{aligned}$$

The two remaining double integrals are transformed in an analogous way. We obtain the result:

$$\begin{aligned} \sigma_{A_k}^2 &= 2 \left\{ \frac{2}{T} \int_0^T \left(1 - \frac{z}{T}\right) m_2(z) \cos \omega_k z \, dz - \right. \\ &\quad \left. - \frac{1}{2\pi k} \cdot \frac{2}{T} \int_0^T m_2(z) \sin \omega_k z \, dz \right\}. \quad (3.49) \end{aligned}$$

Taking a sufficiently long observation time  $T$ , one can make the term  $z/T$  within the brackets of the integrand of the first integral arbitrarily small. Similarly, for all the spectrum components of frequencies  $\omega_k > \alpha$ , where  $\alpha$  is an arbitrarily small positive number, one can reduce at will the role of the second integral of (3.49) by increasing the observation time, since thereby the order  $k$  of the spectrum component increases. Thus, for sufficiently large  $T$  we have:

$$\sigma_{A_k}^2 = 2a_k, \quad (3.50)$$

where  $a_k$  is the amplitude of the corresponding components of the cosine spectrum of the moment  $m_2(z)$ , i.e.,

$$a_k = \frac{2}{T} \int_0^T m_2(z) \cos \omega_k z \, dz. \quad (3.51)$$

Analogously one can find the dispersion of the amplitude  $B_k$  determined by the integral transformation (3.45). If the observation time  $T$  is sufficiently long, the derivations give results coinciding with (3.50):

$$\sigma_{B_k}^2 = 2a_k. \quad (3.52)$$

The spectral component of frequency  $\omega_k$  of a random function can be written as follows:

$$A_k \cos \omega_k t + B_k \sin \omega_k t = C_k \sin(\omega_k t + \varphi_k), \quad (3.53)$$

where

$$C_k^2 = A_k^2 + B_k^2; \quad \varphi_k = \arctg \frac{A_k}{B_k}. \quad (3.54)$$

Since the average of the sum equals always the sum of the average values of summands, we have on the basis of (3.50) and (3.52):

$$\sigma_{C_k}^2 = \sigma_{A_k}^2 + \sigma_{B_k}^2 = 4a_k. \quad (3.55)$$

Relationships (3.50), (3.52) and (3.55) make it possible to make the following statement which is important for the spectral theory of random processes: the spectrum of the mean squares of the amplitudes of a Fourier series, or, in other words, the power spectrum of a stationary random function, coincides (to within a constant factor) with the cosine spectrum of its second moment or correlation function.

Consequently, by analogy with the spectral theory of determinate functions, the conclusion can be drawn that a weakly correlated random function has a broad spectrum while a strongly correlated function has a narrow spectrum. This statement will be given a clear physical interpretation in the following.

Let us continue the investigation of the spectrum of a random function. We shall first clarify the question of statistical dependence between the probability amplitudes  $A_k$  and  $B_k$ . To this end we regard expressions (3.44) and (3.45) as a set of two integral transformations of the random function  $f(t)$  and employ relationship (2.61) to calculate their second-order mixed moment:

$$M[A_k B_k] = \frac{1}{T^2} \int_0^T \int_0^T m_2(x_2 - x_1) \cos \omega_k x_1 \sin \omega_k x_2 \, dx_1 \, dx_2. \quad (3.56)$$

Expression (3.56) can be transformed like (3.46). The result is:

$$M[A_k B_k] = -\frac{4}{T} \int_0^T \frac{z}{T} m_2(z) \sin \omega_k z \, dz. \quad (3.57)$$

In view of equations (3.50), (3.51), (3.52) and (3.57) we note that for  $T \rightarrow \infty$  the correlation coefficient

$$\rho[A_k, B_k] = \frac{M[A_k B_k]}{\sigma_{A_k} \sigma_{B_k}} \quad (3.58)$$

tends to zero, i.e., for a sufficiently long observation time the random amplitudes  $A_k$  and  $B_k$  can be considered as statistically independent.

The question of the statistical dependence between the amplitudes of spectral components of different order, e.g.,  $A_k$  and  $A_m$ ,  $B_k$  and  $B_m$ ,  $A_k$  and  $B_m$  and, finally,  $A_m$  and  $B_k$ , can be investigated in an analogous way. For  $T \rightarrow \infty$  all these amplitudes are statistically independent of each other. Thus, under this condition a random function is resolved into statistically independent components.

We shall turn now to quantitative relationships in the spectrum of a stationary random function. According to (3.55), the mean square of the instantaneous value of the  $k$ -th spectral component can be written as:

$$\sigma_k^2 = \frac{1}{2} \sigma_{C_k}^2 = 2a_k. \quad (3.59)$$

Since the moment  $m_2(z)$  is usually a damped function, the values of  $a_k$  and consequently of  $\sigma_k^2$ , are very small for a long observation time  $T$ . This conclusion follows directly from expression (3.51). However, for large  $T$  the spectral lines lie very close together, and their number in a not too small frequency band is quite considerable.

We shall select from the spectrum of the random function a narrow frequency interval  $\Delta F$  within which the spectrum can be considered as uniform. The number of spectrum components in this interval will be  $\Delta F T$ . Therefore, in view of (3.51) and (3.59) we can express the mean square of the random function in the frequency interval  $\Delta F$  in the following way:

$$\sigma_{\Delta F}^2 = \sigma_k^2 \Delta F T = 2 \Delta F T a_k = 4 \Delta F \int_0^T m_2(z) \cos \omega_k z dz, \quad (3.60)$$

where  $\omega_k$  is the frequency of one of the spectrum components in the interval  $\Delta F$ .

We shall introduce the spectral density  $F(\omega)$  of the random function, interpreting it as the mean square of the random function in a unit interval of the angular frequency. From expression (3.60) we have:

$$F(\omega_k) = \frac{\sigma_{\Delta F}^2}{2\pi \Delta F} = \frac{2}{\pi} \int_0^T m_2(z) \cos \omega_k z dz \quad (3.61)$$

or, going over to the limit for  $T \rightarrow \infty$ , we obtain:

$$F(\omega) = \frac{2}{\pi} \int_0^\infty m_2(z) \cos \omega z dz. \quad (3.62)$$

Thus, the spectral density of a stationary random function is a Fourier transform of its second moment. If the spectral density  $F(\omega)$  of a stationary random function is known, one can find its second moment by using the inverse transformation:

$$m_2(z) = \int_0^\infty F(\omega) \cos \omega z d\omega. \quad (3.63)$$

The mean square of the random function is expressed by its spectral density thus:

$$\sigma^2 = \int_0^\infty F(\omega) d\omega. \quad (3.64)$$

The results obtained, and in the first place the relationships (3.62), (3.63) and (3.64), make it possible to develop a spectral theory of stationary random processes.

Let us examine a system with one input and one output subjected to a stationary external force. It is required to find the steady-state response of the system. As in the spectral theory of dynamic processes it is assumed here that the complex transfer ratio of the system is given:

$$K(j\omega) = \frac{Y(j\omega)}{X(j\omega)}, \quad (3.65)$$

where  $X(j\omega)$  is the complex amplitude of the harmonic applied input,  $Y(j\omega)$  is the complex amplitude of the steady-state harmonic response of the system.

Knowing the spectral density  $F_{in}(\omega)$  of the applied input or having calculated it with the aid of expression (3.62), and being given the second moment of the response one can calculate the spectral density  $F_{out}(\omega)$  of the system's response

$$F_{out}(\omega) = F_{in}(\omega) |K(j\omega)|^2. \quad (3.66)$$

If the spectral density of the system's response is known, expression (3.63) makes it possible to calculate its second moment:

$$m_2^{(y)}(z) = \int_0^\infty F_{out}(\omega) \cos z\omega d\omega \quad (3.67)$$

and, in particular, its mean square

$$\sigma_y^2 = m_2^{(y)}(0) = \int_0^\infty F_{out}(\omega) d\omega. \quad (3.68)$$

The above considerations have been elaborated independently of the results of the preceding section obtained in working out the method of impulse characteristics. Another approach is possible in which one starts from the relationships of the preceding section and bases the spectral method upon them. We shall examine this approach dealing, as yet, only with systems which have one input and one output.

In the preceding section we obtained formula (3.30) for the second moment at the output of the system. Taking into account the stationary nature of the processes dealt with we shall rewrite this formula, taking  $-\infty$  as the lower limit

$$m_2^{(y)}(t_1, t_2) = \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} m_2^{(x)}(t'_1, t'_2) \xi(t_1 - t'_1) \xi(t_2 - t'_2) dt'_1 dt'_2. \quad (3.69)$$

We can obtain from expression (3.69) the above-shown basic relationships of the spectral method. Before proving this we shall apply further transformations to (3.69) using the property of stationarity of the process considered.

In stationary processes the second moments appearing in (3.69) are even functions of the differences  $t_2 - t_1$ , and  $t'_2 - t'_1$ . Further, since only the relative position of the time instants  $t_1$  and  $t_2$  is important, one can assume  $t_1 = 0$  without loss of generality. Now, introducing the notation  $\tau = t_2 - t_1 = t_2$  and changing the integration variables:  $t'_1 = -\theta_1, t'_2 = -\theta_2$ , we can write (3.69) in the form:

$$m_2^{(y)}(z) = \int_{-z}^{\infty} \xi(\theta_2 + z) d\theta_2 \int_{\theta_2}^{\infty} m_2^{(x)}(\theta_1 - \theta_2) \xi(\theta_1) d\theta_1, \quad (3.70)$$

or, if we set  $\theta_1 - \theta_2 = \theta$ ,

$$m_2^{(y)}(z) = \int_{-z}^{\infty} \xi(\theta_2 + z) d\theta_2 \int_{-\theta_2}^{\infty} m_2^{(x)}(\theta) \xi(\theta + \theta_2) d\theta. \quad (3.71)$$

Let us change the order of integration in the double integral of (3.71). Then we shall obtain:

$$\begin{aligned} m_2^{(y)}(z) = & \int_{-\infty}^z m_2^{(x)}(\theta) d\theta \int_{-\theta}^{\infty} \xi(\theta + \theta_2) \xi(\theta_2 + z) d\theta_2 + \\ & + \int_z^{\infty} m_2^{(x)}(\theta) d\theta \int_{-z}^{\infty} \xi(\theta + \theta_2) \xi(\theta_2 + z) d\theta_2. \end{aligned} \quad (3.72)$$

In the inner integral of the first term (3.72) the integrand vanishes for  $\theta_2 < -\theta$ . Therefore the lower limit of this integral can be formally replaced by  $-\infty$ . A similar change is equally possible in the inner integral of the second term since for  $\theta_2 < -z$  we have  $\xi(\theta_2 + z) = 0$ . Consequently,

$$\begin{aligned} m_2^{(y)}(z) = & \int_{-\infty}^z m_2^{(x)}(\theta) d\theta \int_{-\infty}^{+\infty} \xi(\theta + \theta_2) \xi(\theta_2 + z) d\theta_2 + \\ & + \int_z^{\infty} m_2^{(x)}(\theta) d\theta \int_{-\infty}^{+\infty} \xi(\theta + \theta_2) \xi(\theta_2 + z) d\theta_2 = \\ = & \int_{-\infty}^{+\infty} m_2^{(x)}(\theta) d\theta \int_{-\infty}^{+\infty} \xi(\theta + \theta_2) \xi(\theta_2 + z) d\theta_2. \end{aligned} \quad (3.73)$$

carrying out the substitution  $\theta_2 + \theta = \psi$ , in the inner integral of (3.73) we obtain:

$$\begin{aligned} m_2^{(y)}(z) = & \int_{-\infty}^{+\infty} m_2^{(x)}(\theta) d\theta \int_{-\infty}^{+\infty} \xi(\psi) \xi(\psi - \theta + z) d\psi = \\ = & \int_{-\infty}^{+\infty} m_2^{(x)}(\theta) \varphi(\theta, z) d\theta, \end{aligned} \quad (3.74)$$

where

$$\varphi(\theta, z) = \int_{-\infty}^{+\infty} \xi(\psi) \xi(\psi - \theta + z) d\psi. \quad (3.75)$$

We have now completed the preliminary transformation of expression (3.69) and proceed to the basic exposition of the spectral method.

As well-known [11] the complex transfer ratio of the linear system and its impulse characteristic are connected by the direct and inverse Fourier transforms:

$$K(j\omega) = \int_{-\infty}^{+\infty} \xi(t) e^{-j\omega t} dt, \quad (3.76)$$

$$\xi(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} K(j\omega) e^{j\omega t} d\omega. \quad (3.77)$$

Furthermore, the following relation is known in the theory of Fourier transforms:

$$\int_{-\infty}^{+\infty} f(t)f(t+\tau)dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |f_{\omega}|^2 e^{j\omega\tau} d\omega, \quad (3.78)$$

where  $f_{\omega}$  is the spectral density of the function  $f(t)$ :

$$f_{\omega} = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt. \quad (3.79)$$

Making use of these results we can rewrite equation (3.75) thus:

$$\varphi(\theta, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |K(j\omega)|^2 e^{j\omega(z-\theta)} d\omega. \quad (3.80)$$

Substituting this expression for  $\varphi(\theta, z)$  in (3.74) and changing the order of integration we obtain:

$$\begin{aligned} m_2^{(y)}(z) &= \int_{-\infty}^{+\infty} m_2^{(x)}(\theta) d\theta \cdot \frac{1}{2\pi} \int_{-\infty}^{+\infty} |K(j\omega)|^2 e^{j\omega z} e^{-j\omega\theta} d\omega = \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} |K(j\omega)|^2 e^{j\omega z} d\omega \cdot \frac{1}{\pi} \int_{-\infty}^{+\infty} m_2^{(x)}(\theta) e^{-j\omega\theta} d\theta. \end{aligned} \quad (3.81)$$

Since the second moment  $m_2^{(x)}(\theta)$  of the applied input is even, then, taking into account (3.62), the inner integral of expression (3.81) can be transformed in the following way:

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} m_2^{(x)}(\theta) e^{-j\omega\theta} d\theta = \frac{2}{\pi} \int_0^{\infty} m_2^{(x)}(\theta) \cos \omega\theta d\theta. \quad (3.82)$$

The integral of (3.82) is an even function of the frequency  $\omega$ . Similarly, the square of the modulus of the system's transfer ratio  $|K(j\omega)|^2$  is an even function of the frequency. This makes it possible to write expression (3.81) in the following final form:

$$m_2^{(y)}(z) = \int_0^{\infty} |K(j\omega)|^2 \cos \omega z d\omega \cdot \frac{2}{\pi} \int_0^{\infty} m_2^{(x)}(\theta) \cos \omega\theta d\theta. \quad (3.83)$$

This result is entirely equivalent to the one obtained earlier for the relationships of the spectral method. In fact, comparing integral (3.82) with expression (3.62) we notice that the integral gives the spectral density of the applied input. Further, taking into account expression (3.66) we see that (3.83) has the same meaning as the earlier result (3.67).

The difference between the second way of establishing the spectral method and the first one consists in the fact that in the first one the concept of the spectrum of the random function itself was introduced while in the second one appears the spectrum of only its correlation ratio. Therefore, the second one leads us more quickly to the goal. However, in some cases the spectrum of the random function is a very useful concept.

Let us proceed now to linear systems with several inputs and outputs. For the sake of simplicity we shall examine here, as hitherto, systems with two inputs and two outputs. The generalization of the spectral method which we have in mind can be achieved by any one of the two ways given above. Here we give preference to the second one. This is not due to basic considerations but is explained by the desire to shorten the calculations necessary for obtaining the final results.

Our system will be characterized by four transfer ratios:

$|K_{11}(j\omega)|$ ,  $|K_{12}(j\omega)|$ ,  $|K_{21}(j\omega)|$  and  $|K_{22}(j\omega)|$ , and also by the four phase shifts:  $\varphi_{11}(\omega)$ ,  $\varphi_{12}(\omega)$ ,  $\varphi_{21}(\omega)$  and  $\varphi_{22}(\omega)$ , where the first and second subscripts refer to the output and the input, respectively.

In the preceding section we obtained expression (3.39) for the second moment at the first output. Taking into account the stationary nature of the processes considered, this expression can be written as follows:

$$\begin{aligned} m_2^{(y)}(t_1, t_2) = & \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} m_2^{(x_1)}(t'_1, t'_2) \xi_{11}(t_1 - t'_1) \xi_{11}(t_2 - t'_2) dt'_1 dt'_2 + \\ & + \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} m_2^{(x_1, x_2)}(t'_1, t'_2) \xi_{11}(t_2 - t'_2) \xi_{12}(t_1 - t'_1) dt'_1 dt'_2 + \\ & + \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} m_2^{(x_2, x_1)}(t'_1, t'_2) \xi_{11}(t_1 - t'_1) \xi_{12}(t_2 - t'_2) dt'_1 dt'_2 + \\ & + \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} m_2^{(x_2)}(t'_1, t'_2) \xi_{12}(t_1 - t'_1) \xi_{12}(t_2 - t'_2) dt'_1 dt'_2. \end{aligned} \quad (3.84)$$

The first and the fourth integrals of (3.84) have the same form as the integral of (3.69). Therefore, by analogy with (3.83) one can write:

$$\begin{aligned} \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} m_2^{(x_1)}(t'_1, t'_2) \xi_{11}(t_1 - t'_1) \xi_{11}(t_2 - t'_2) dt'_1 dt'_2 = & \quad (3.85) \\ = \int_0^\infty |K_{11}(j\omega)|^2 \cos \omega z dz \cdot \frac{2}{\pi} \int_0^\infty m_2^{(x_1)}(\theta) \cos \omega \theta d\theta, \\ \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} m_2^{(x_2)}(t'_1, t'_2) \xi_{12}(t_1 - t'_1) \xi_{12}(t_2 - t'_2) dt'_1 dt'_2 = & \\ = \int_0^\infty |K_{12}(j\omega)|^2 \cos \omega z d\omega \cdot \frac{2}{\pi} \int_0^\infty m_2^{(x_2)}(\theta) \cos \omega \theta d\theta. \end{aligned} \quad (3.86)$$

The integral of (3.85) gives the second moment of the response that would have taken place if external force were applied at the first input only. Integral (3.86) is a response analogous to the characteristic for external force applied at the second input only.

We shall proceed to the transformation of the second integral of (3.84). Unlike equation (3.69), where the function  $m_2^{(x)}(t'_1, t'_2)$  was an even function of the difference  $t'_2 - t'_1$ , here the function  $m_2^{(x_1, x_2)}(t'_1, t'_2)$  although being a function of the difference  $t'_2 - t'_1$  (by virtue of the stationary nature of the process), is not in general an even function. Since in the course of the transformations which led us from equation (3.69) to (3.74) and (3.75) we had not assumed that the function is even, we need not repeat the above transformation and we can write directly:

$$\begin{aligned} \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} m_2^{(x_1, x_2)}(t'_1, t'_2) \xi_{11}(t_2 - t'_2) \xi_{12}(t_1 - t'_1) dt'_1 dt'_2 = \\ = \int_{-\infty}^{+\infty} m_2^{(x_1, x_2)}(\theta) \varphi(\theta, z) d\theta, \end{aligned} \quad (3.87)$$

where

$$\varphi(\theta, z) = \int_{-\infty}^{+\infty} \xi_{12}(\psi) \xi_{11}(\psi - \theta + z) d\psi. \quad (3.88)$$

For the further transformation of expressions (3.87) and (3.88) we shall use the relationship

$$\int_{-\infty}^{+\infty} f(t) g(t + \tau) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f_{\omega}^* g_{\omega} e^{j\omega\tau} d\omega, \quad (3.89)$$

which is known from the theory of Fourier integrals, and in which  $f_{\omega}$  and  $g_{\omega}$  are the spectral densities of the functions  $f(t)$  and  $g(t)$  and in which the asterisk indicates the complex conjugate.

Since, according to expression (3.76), the system's transfer ratio is given by the Fourier transform of its impulse characteristic and in view of equation (3.89), we can rewrite equation (3.88) as follows:

$$\varphi(\theta, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} K_{12}^*(j\omega) K_{11}(j\omega) e^{j\omega(\theta - z)} d\omega. \quad (3.90)$$

We now substitute the obtained value of  $\varphi(\theta, z)$  in (3.87) and, changing the order of integration, we have:

$$\begin{aligned} & \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} m_2^{(x_1, x_2)}(t'_1, t'_2) \xi_{11}(t_2 - t'_2) \xi_{12}(t_1 - t'_1) dt'_1 dt'_2 = \\ & = \frac{1}{2} \int_{-\infty}^{+\infty} K_{12}^*(j\omega) K_{11}(j\omega) e^{j\omega z} d\omega \cdot \frac{1}{\pi} \int_{-\infty}^{+\infty} m_2^{(x_1, x_2)}(\theta) e^{-j\omega\theta} d\theta. \end{aligned} \quad (3.91)$$

An analogous form can be given to the third integral of expression (3.84):

$$\begin{aligned} & \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} m_2^{(x_2, x_1)}(t'_1, t'_2) \xi_{11}(t_1 - t'_1) \xi_{12}(t_2 - t'_2) dt'_1 dt'_2 = \\ & = \frac{1}{2} \int_{-\infty}^{+\infty} K_{11}^*(j\omega) K_{12}(j\omega) e^{j\omega z} d\omega \cdot \frac{1}{\pi} \int_{-\infty}^{+\infty} m_2^{(x_2, x_1)}(\theta) e^{-j\omega\theta} d\theta. \end{aligned} \quad (3.92)$$

Now, using the results of (3.85), (3.86), (3.91) and (3.92) we rewrite (3.84) as follows:

$$\begin{aligned} m_2^{(y_1)}(z) &= \int_0^{\infty} |K_{11}(j\omega)|^2 \cos \omega z d\omega \cdot \frac{2}{\pi} \int_0^{\infty} m_2^{(x_1)}(\theta) \cos \omega \theta d\theta + \\ &+ \int_0^{\infty} |K_{12}(j\omega)|^2 \cos \omega z d\omega \cdot \frac{2}{\pi} \int_0^{\infty} m_2^{(x_2)}(\theta) \cos \omega \theta d\theta + \\ &+ \frac{1}{2} \int_{-\infty}^{+\infty} [K_{11}(j\omega) K_{12}^*(j\omega) + K_{11}^*(j\omega) \cdot K_{12}(j\omega)] e^{j\omega z} d\omega \times \\ &\times \frac{1}{\pi} \int_{-\infty}^{+\infty} m_2^{(x_1, x_2)}(\theta) e^{-j\omega\theta} d\theta. \end{aligned} \quad (3.93)$$

Let us analyze the equation obtained. Comparing the first integral of (3.93) with equation (3.83) we conclude that this integral gives the second moment of the response which would take place if the external force were applied at the first input only. The internal integral of the first term multiplied by  $2/\pi$  gives, as follows from (3.62), the spectral density of external force at the first input:

$$\frac{2}{\pi} \int_0^{\infty} m_2^{(x_1)}(\theta) \cos \omega \theta d\theta = F_{x_1}(\omega). \quad (3.94)$$

The second integral expresses the second moment of the response under external force at the second input only. Similarly to (3.94) we have:

$$\frac{2}{\pi} \int_0^{\infty} m_2^{(x_2)}(\theta) \cos \omega \theta d\theta = F_{x_2}(\omega). \quad (3.95)$$

The third integral is the result of the statistical dependence between the external force at the two inputs. The inner integral of the third term with a factor of  $1/\pi$

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} m_2^{(x_1, x_2)}(\theta) e^{-j\omega\theta} d\theta = F_{x_1, x_2}(j\omega) \quad (3.96)$$

expresses the spectral density of the mixed moment  $m_2^{(x_1, x_2)}(\theta)$ . We shall call this spectral density the mutual spectral density of the external forces. Since in general the mixed moment is not an even function, transformation (3.82) is not applicable here and the mutual spectral density itself is complex.

Taking the aforesaid into account, we can write equation (3.93) in the following more compact form:

$$\begin{aligned} m_2^{(y)}(z) = & \int_0^{\infty} |K_{11}(j\omega)|^2 F_{x_1}(\omega) \cos \omega z d\omega + \\ & + \int_0^{\infty} |K_{12}(j\omega)|^2 F_{x_2}(\omega) \cos \omega z d\omega + \\ & + \frac{1}{2} \int_{-\infty}^{+\infty} [K_{11}(j\omega) K_{12}^*(j\omega) + K_{11}^*(j\omega) K_{12}(j\omega)] F_{x_1, x_2}(j\omega) e^{j\omega z} d\omega. \end{aligned} \quad (3.97)$$

An analogous relationship is valid for the second moment of response at the second output:

$$\begin{aligned} m_2^{(y)}(z) = & \int_0^{\infty} |K_{21}(j\omega)|^2 F_{x_1}(\omega) \cos \omega z d\omega + \\ & + \int_0^{\infty} |K_{22}(j\omega)|^2 F_{x_2}(\omega) \cos \omega z d\omega + \\ & + \frac{1}{2} \int_{-\infty}^{+\infty} [K_{21}(j\omega) K_{22}^*(j\omega) + K_{21}^*(j\omega) K_{22}(j\omega)] F_{x_1, x_2}(j\omega) e^{j\omega z} d\omega. \end{aligned} \quad (3.98)$$

To calculate the mixed moment of response at both outputs of the system we use equation (3.41) of the preceding section which assumes the form

$$\begin{aligned} m_2^{(y_1, y_2)}(t_1, t_2) = & \int_{-\infty}^{t_1} \int_{-\infty}^{t_1} m_2^{(r)}(t'_1, t'_2) \xi_{11}(t_1 - t'_1) \xi_{21}(t_2 - t'_2) dt'_1 dt'_2 + \\ & + \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} m_2^{(x_1, x_2)}(t'_1, t'_2) \xi_{12}(t_1 - t'_1) \xi_{21}(t_2 - t'_2) dt'_1 dt'_2 + \\ & + \int_{-\infty}^{t_1} \int_{-\infty}^{t_1} m_2^{(x_1, x_2)}(t'_1, t'_2) \xi_{11}(t_1 - t'_1) \xi_{22}(t_2 - t'_2) dt'_1 dt'_2 + \\ & + \int_{-\infty}^{t_2} \int_{-\infty}^{t_2} m_2^{(r)}(t'_1, t'_2) \xi_{12}(t_1 - t'_1) \xi_{22}(t_2 - t'_2) dt'_1 dt'_2. \end{aligned} \quad (3.99)$$

for stationary processes.

The transformation of analogous integrals had been carried out in the

foregoing. Therefore, without writing out the corresponding derivations, we shall give immediately their final result:

$$m_2^{(u, v)}(z) = \frac{1}{2} \int_{-\infty}^{+\infty} [K_{11}^*(j\omega) K_{21}(j\omega) F_{x_1}(\omega) + K_{12}^*(j\omega) K_{22}(j\omega) F_{x_2}(\omega) + \{K_{11}^*(j\omega) K_{22}(j\omega) + K_{12}^*(j\omega) K_{21}(j\omega)\} F_{x_1 x_2}(j\omega)] e^{j\omega z} d\omega. \quad (3.100)$$

It is understood here that the order of the two time instants is such that  $t_2 > t_1$ .

It is easy to see that the square brackets in the integrand of (3.100) contain the mutual spectral density of response at both outputs.

Thus, Equations (3.97), (3.98) and (3.100) obviously determine the statistical properties of the responses at both outputs of the system.

#### § 15. An RC Circuit excited by a Stationary Fluctuating Voltage

In this section we are concerned with application of methods expounded in the preceding sections to a simple particular problem. This problem is given here as a clear physical illustration, being at the same time the point of departure for some generalizations to be given later.

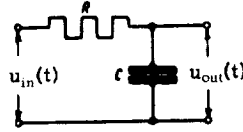


Figure 2. An RC Circuit excited by a fluctuating voltage

Let an electric fluctuating voltage  $u_{in}(t)$  be applied at the input across the RC circuit (Figure 2) at the moment  $t = 0$ . It is assumed that this voltage is a stationary random function of time, whereby its d.c. component (first-order moment) equals zero, and the second-order moment, which in this case coincides with the correlation ratio is

$$m_2^{(in)}(t_1, t_2) = \sigma_{in}^2 e^{-\beta |t_2 - t_1|}. \quad (3.101)$$

It is easy to see that the quantity  $\sigma_{in}^2$  is the square of the effective value of the input voltage. The initial charge on the capacitor is assumed equal to zero.

It is necessary to find the steady-state law of the second moment of output voltage. This will make it possible, in particular, to find the law of the variation of the effective fluctuating voltage at the output.

We shall use the method of stochastic differential equations for the solution of this problem. The stochastic differential equation which describes the investigated random process has the form

$$\frac{du_{out}(t)}{dt} + \alpha u_{out}(t) = \alpha u_{in}(t), \quad (3.102)$$

where  $\alpha = 1/RC$

Thus, the operators  $A_t$  and  $B_t$  of equation (3.1) are in this case expressed by

$$A_t = \frac{d}{dt} + a; \quad B_t = a. \quad (3.103)$$

Therefore, the differential equation (3.8) reads in this particular case

$$\left(\frac{\partial}{\partial t_1} + a\right) \left(\frac{\partial}{\partial t_2} + a\right) m_2^{(out)}(t_1, t_2) = a^2 m_2^{(in)}(t_1, t_2). \quad (3.104)$$

To represent the required second moment in the form of (3.21) it is necessary to find the double transform of the second moment of the input, determined by expression (3.101). Let us calculate its Laplace transform to the variable  $t_1$ :

$$\begin{aligned} \bar{m}_2^{(in)}(p_1, t_2) &= \int_0^\infty m_2^{(in)}(t_1, t_2) e^{-p_1 t_1} dt_1 = \\ &= a_{in}^2 \left\{ \int_0^{t_2} e^{-\beta(t_2-t_1)} e^{-p_1 t_1} dt_1 + \int_{t_2}^\infty e^{-\beta(t_1-t_2)} e^{-p_1 t_1} dt_1 \right\} = \\ &= \frac{a_{in}^2}{\beta^2 - p_1^2} \{ 2\beta e^{-p_1 t_2} - (\beta + p_1) e^{-\beta t_2} \}. \end{aligned} \quad (3.105)$$

Now we shall transform the obtained expression with respect to the variable  $t_2$ :

$$\begin{aligned} \bar{m}_2^{(in)}(p_1, p_2) &= \int_0^\infty \bar{m}_2^{(in)}(p_1, t_2) e^{-p_2 t_2} dt_2 = \\ &= \frac{a_{in}^2}{\beta^2 - p_1^2} \left\{ 2\beta \int_0^\infty e^{-(p_1 + p_2) t_2} dt_2 - (\beta + p_1) \int_0^\infty e^{-(p_1 + \beta) t_2} dt_2 \right\} = \\ &= a_{in}^2 \frac{2\beta + p_1 + p_2}{(p_1 + \beta)(p_2 + \beta)(p_1 + p_2)}. \end{aligned} \quad (3.106)$$

The transition from the operators  $A_t$  and  $B_t$  to their transforms gives, according to (3.13) and (3.15):

$$\bar{A}_p = p + a; \quad \bar{B}_p = a. \quad (3.107)$$

Therefore, the required double transform of the second moment which has in general the form of (3.21) is obtained in this case as

$$\bar{m}_2^{(out)}(p_1, p_2) = a^2 a_{in}^2 \frac{2\beta + p_1 + p_2}{(p_1 + a)(p_1 + \beta)(p_1 + p_2)(p_2 + a)(p_2 + \beta)}. \quad (3.108)$$

The inverse double transition to the original, which can easily be carried out with the aid of a table of operational relations, leads to the required result:

$$\begin{aligned} m_2^{(out)}(t_1, t_2) &= \frac{a^2 a_{in}^2}{a^2 - \beta^2} \left\{ a e^{-\beta |t_2 - t_1|} - \beta e^{-a |t_2 - t_1|} + \right. \\ &\quad \left. + (a + \beta) e^{-a(t_1 + t_2)} - a [e^{-(\beta t_1 + a t_2)} + e^{-(a t_1 + \beta t_2)}] \right\}. \end{aligned} \quad (3.109)$$

Let us examine the obtained expression. As should have been expected, it is symmetric with respect to the variables  $t_1$  and  $t_2$  and satisfies the initial condition  $m_2^{(out)}(0, 0) = 0$ . The second moment of the response depends, not only upon the absolute value of the difference  $|t_2 - t_1|$ , but also upon the disposition of both time instants with respect to the time the input is switched on. Thus, there is a nonstationary random process at the output of the circuit. The steady-state

law of the mean square of the output voltage is obtained by setting in (3.109)

$t_1 = t_2 = t$ :

$$\sigma_{\text{out}}^2(t) = \frac{a}{a+\beta} \sigma_{\text{in}}^2 \left\{ 1 - \frac{e^{-at}}{a-\beta} (ae^{-\beta t} - \beta e^{-at}) \right\}. \quad (3.110)$$

If  $t_1 \rightarrow \infty$  and  $t_2 \rightarrow \infty$  while the absolute value of their difference  $|t_2 - t_1|$  remains finite, the second moment of the output voltage tends to its steady value:

$$m_{st}^{(\text{out})}(t_1, t_2) = \frac{\sigma_{\text{in}}^2}{a^2 - \beta^2} [ae^{-\beta|t_2 - t_1|} - \beta e^{-a|t_2 - t_1|}]. \quad (3.111)$$

Now the second moment of the response depends only upon the relative position of the time instants  $t_1$  and  $t_2$ , i.e., in the limit a stationary random process is obtained.

Assuming in (3.110)  $t \rightarrow \infty$  or in (3.111)  $|t_2 - t_1| = 0$ , we find the mean square of the fluctuation steady-state voltage at the output of the circuit:

$$\sigma_{\text{out}}^2 = \sigma_{\text{in}}^2 \frac{a}{a+\beta}. \quad (3.112)$$

The normalized steady-state correlation ratio will be:

$$\rho_{\text{out}}(t_1, t_2) = \frac{m_{st}^{(\text{out})}(t_1, t_2)}{\sigma_{\text{out}}^2} = \frac{\sigma e^{-\beta|t_2 - t_1|} - \beta e^{-a|t_2 - t_1|}}{a - \beta}. \quad (3.113)$$

Thus, the necessary quantitative relations have been obtained. Let us analyze them.

We shall call correlation period the time interval within which there is a significant statistical dependence between the values of the random function (voltage in our case). As a criterion of this dependence one may take  $\rho(|t_1 - t_2|) \geq 0.1$ , for instance. Furthermore, let us call the time in which the voltage of the capacitor is reduced to 10% of its initial value, the discharge period of the capacitor. Then equating the correlation period of the input voltage to the discharge period of the capacitor corresponds to equating  $\alpha$  and  $\beta$ .

Let  $\beta \ll \alpha$ , i.e., the correlation period of the applied input greatly exceeds the discharge period of the capacitor. In other words, the speed at which the random process occurs in the circuit is much greater than the mean rate of change of the input voltage. In this case we obtain from (3.112) and (3.113):

$$\sigma_{\text{out}}^2 \approx \sigma_{\text{in}}^2; \quad \rho_{\text{out}}(t_1, t_2) \approx e^{-\beta|t_2 - t_1|}. \quad (3.114)$$

Equation (3.114) signifies that under the indicated conditions the statistical properties of the output voltage coincide with these of the input voltage. The reason for this lies in the fact for  $\beta \ll \alpha$  the voltage at the output of the circuit manages to follow the variations in the input.

We shall assume now that  $\alpha \ll \beta$ , which corresponds to a long discharge period of the capacitor as compared with the correlation period of the input voltage. Then equation (3.113) reduces to

$$\rho_{\text{out}}(t_1, t_2) = e^{-a|t_2 - t_1|}, \quad (3.115)$$

i.e., the degree of statistical dependence at the output does not depend upon such dependence at the input, but is determined exclusively by the parameters of the circuit.

It is noteworthy that with weak correlation of the applied input, ( $\alpha \ll \beta$ ), the equation of the correlation ratio of the output voltage has the same form as the equation of the discharge of a capacitor through a resistance i.e., the equation of the specific transient process of the system. This correspondence is not accidental. The reason for it will be explained in the next section.

Let us note that when speaking of a weak or strong correlation of the input we should not consider its correlation period unrelated, but we ought to compare the correlation period with the duration of the specific transient process of the system to which the excitation is applied.

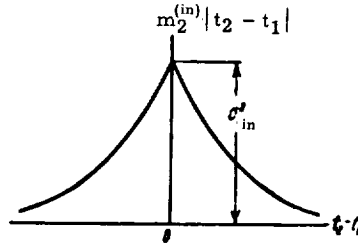


Figure 3. Graph of the second moment of the voltage across the input of an RC circuit

Let us explain yet another characteristic property of a weakly correlated input. For this we shall turn to expression (3.112) which, for  $\alpha \ll \beta$  can be written thus:

$$\sigma_{out}^2 = \alpha \frac{\sigma_{in}^2}{\beta}. \quad (3.116)$$

For a clear interpretation of the relation obtained let us compute the area  $S$  bounded by the graph of the input moment (3.101) and the abscissa (Figure 3). We shall put there  $|t_2 - t_1| = \tau$ . Then

$$S = 2 \int_0^{\infty} \sigma_{in}^2 e^{-\beta \tau} d\tau = 2 \frac{\sigma_{in}^2}{\beta}. \quad (3.117)$$

taking the obtained result into account we shall rewrite expression (3.116) in the following way:

$$\sigma_{out}^2 = \frac{1}{2} \alpha S. \quad (3.118)$$

Thus, the mean square of the system's response to a weakly correlated input is determined by the area  $S$ , which characterizes this input and by the parameters of the system. The present result was obtained for the specific system with the particular form (3.101) of the function  $m_2^{(in)}(t_1, t_2)$ . It is shown in the following section that this statement is of general application.

All the results listed above have been obtained by the method of stochastic differential equations. Let us solve the same problem by the method of impulse characteristics. To make derivations brief, we shall compute the second moment of the response only in the stationary regime. For this we shall use expression (3.30) in which the lower integration limits are taken equal to  $-\infty$ .

We shall assume for the sake of definiteness  $t_1 > t_2$  and we shall calculate

the inner integral of expression (3.30) with the above-mentioned change of the lower limit. The impulse characteristic of an RC circuit  $\xi$ , as well known, given by:

$$\xi(t) = \alpha e^{-\alpha t}, \quad (3.119)$$

Taking into account relation (3.101) for the second moment of the applied input we have:

$$\begin{aligned} \int_{-\infty}^{t_1} m_2^{(x)}(t'_1, t'_2) \xi(t_1 - t'_1) dt'_1 = \\ = \alpha \sigma_{in}^2 \left\{ \int_{-\infty}^{t'_2} e^{-\beta(t'_2 - t'_1)} e^{-\alpha(t_1 - t'_1)} dt'_1 + \right. \\ \left. + \int_{t'_2}^{t_1} e^{-\beta(t'_1 - t'_2)} e^{-\alpha(t_1 - t'_1)} dt'_1 \right\} = \\ = \alpha \sigma_{in}^2 \left\{ \frac{e^{-\beta t_1}}{\alpha - \beta} e^{\beta t'_2} - \frac{2\beta e^{-\alpha t_1}}{\alpha^2 - \beta^2} e^{\alpha t'_2} \right\}. \end{aligned} \quad (3.120)$$

According to (3.30), the result obtained should be multiplied by  $\alpha e^{-\alpha(t_1 - t'_2)}$  and integrated with respect to  $t_2$  between the limits  $-\infty$  and  $t_2$ . Carrying out the said calculations we finally obtain:

$$m_{2\pi}^{(out)}(t_1, t_2) = \frac{\alpha \sigma_{in}^2}{\alpha^2 - \beta^2} [\alpha e^{-\beta(t_1 - t_2)} - \beta e^{-\alpha(t_1 - t_2)}], \quad (3.121)$$

i.e., an expression which coincides with (3.111).

Let us carry out the same calculations by the spectral method. First, using (3.62) we shall find the spectral density of the applied input

$$F_{in}(\omega) = \frac{2}{\pi} \int_0^\infty \sigma_{in}^2 e^{-\beta z} \cos \omega z dz = \frac{2}{\pi} \sigma_{in}^2 \frac{\beta}{\beta^2 + \omega^2}. \quad (3.122)$$

The transfer ratio of the circuit equals:

$$|K(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} = \frac{\alpha}{\sqrt{\alpha^2 + \omega^2}}. \quad (3.123)$$

Therefore the spectral density of its response is, according to (3.66):

$$F_{out}(\omega) = \frac{2}{\pi} \sigma_{in}^2 \frac{\alpha^2 \beta}{(\alpha^2 + \omega^2)(\beta^2 + \omega^2)}. \quad (3.124)$$

Now, using expression (3.63) we find the second moment of the steady-state response of the circuit:

$$\begin{aligned} m_2^{(out)}(z) &= \int_0^\infty \frac{2}{\pi} \sigma_{in}^2 \frac{\alpha^2 \beta \cos z\omega}{(\alpha^2 + \omega^2)(\beta^2 + \omega^2)} d\omega = \\ &= \frac{\alpha \sigma_{in}^2}{\alpha^2 - \beta^2} (\alpha e^{-\beta z} - \beta e^{-\alpha z}). \end{aligned} \quad (3.125)$$

This result had been obtained earlier by the method of stochastic differential equations (formula 3.111) and by the method of impulse characteristics (formula 3.121).

## § 16. Uncorrelated Input

In many problems of the theory of random processes the input is weakly correlated. In this case it is both possible and expedient to idealize the properties of the input. This idealization leads to the concept of  $\delta$ -correlated or uncorrelated input.

We shall use the method of impulse characteristics to carry out the mentioned idealization. We shall turn to the general expression (3.30) and subject it to a transformation for the case of weak correlation of the applied input. Let us examine the inner integral of this expression, where we set  $t_2 > t_1$ .

Weak correlation of the input signifies that the major part of the area  $S$ , bounded by the graph of second moment  $m_2^{(x)}(t'_1, t'_2)$  and the abscissa, spans a narrow interval of the values of  $t'_1$  which contains the time moment  $t'_2$ . The narrowness of the mentioned interval should be understood in the sense, that the variation of the function  $\xi(t'_1 - t'_1)$  in this interval is insignificant, and it can be replaced in this interval by its value at the point  $t'_2$ , i.e., by the constant  $\xi(t'_1 - t'_2)$ . This makes it possible to replace the inner integral by the following approximate expression:

$$\int_0^{t_1} m_2^{(x)}(t'_1, t'_2) \xi(t_1 - t'_1) dt'_1 \approx \xi(t_1 - t'_2) \int_0^{t_1} m_2^{(x)}(t'_1, t'_2) dt'_1 = \quad (3.126)$$

$$= S \xi(t_1 - t'_2),$$

where the area  $S$  is in the general case a function of  $t'_2$ . The weaker the correlation of the input, the more exact is the estimate (3.126). Formula (3.30) now assumes the form

$$m_2^{(y)}(t_1, t_2) \approx \int_0^{t_1} S(t'_2) \xi(t_1 - t'_2) \xi(t_2 - t'_2) dt'_2. \quad (3.127)$$

The result obtained confirms the statement of the preceding section (following formula (3.118)) that, when a weakly correlated random input acts upon a system, the area  $S$  fully characterizes the random process.

Since the form of the boundary of  $S$  is irrelevant under the conditions considered, the actual moment  $m_2^{(x)}(t'_1, t'_2)$  may be replaced by an impulse function of equal area, i.e.,

$$m_2^{(x)}(t'_1, t'_2) = S \delta(t'_2 - t'_1), \quad (3.128)$$

where  $\delta(t'_2 - t'_1)$  is a unit impulse.

An analogous situation exists in the theory of determinate transient processes. If the system is acted upon by an impulse the duration of which is much shorter than the duration of transient processes in the system, such an impulse may be replaced by an impulse function, i.e., an impulse equal in area to the actual one, having infinitely small width and infinite height.

An input whose second moment is of the form of (3.128) is called  $\delta$ -correlated or uncorrelated. The estimate (3.127) becomes exact for such an input:

$$m_2^{(y)} = \int_0^{t_1} S(t'_2) \xi(t_1 - t'_2) \xi(t_2 - t'_2) dt'_2. \quad (3.129)$$

In the particular case when the applied input is stationary we have:

$$m_2^{(y)}(t_1, t_2) = S \int_0^{t_1} \xi(t_1 - t'_1) \xi(t_2 - t'_2) dt'_2. \quad (3.130)$$

If the steady-state second moment of the system's response is sought, the lower limit of the integral of (3.130) should be replaced by  $-\infty$ , i.e.,

$$m_2^{(y)}(t_1, t_2) = S \int_{-\infty}^{t_1} \xi(t_1 - t'_1) \xi(t_2 - t'_2) dt'_2. \quad (3.131)$$

The last equation can be presented in a form which is more convenient for computations. We shall set  $t_1 - t_2 = \tau$  and introduce a new integration variable  $x = t_2 - t'_2$ . Then we obtain:

$$m_2^{(y)}(\tau) = S \int_0^{\infty} \xi(x) \xi(x + \tau) dx. \quad (3.132)$$

The last equation gives the connection between the steady-state second moment of response and the impulse characteristic of the system if the input is uncorrelated.

Other relationships of the method of impulse characteristics are easily transformed in a way analogous to the above.

Let us see how the general relations of the method of stochastic differential equations become simplified for uncorrelated input. For this kind of input its moment in differential equation (3.8) which connects the second moments of output and input should be replaced by the value of the moment from (3.128). In accordance with this, let us calculate the double transform of the second moment of the input, entering in expression (3.21). We shall assume here the input as stationary. The transformation with respect to the variable  $t_1$  gives:

$$\overline{m_2^{(x)}}(\rho_1, t_2) = \int_0^{\infty} S \delta(t_2 - t_1) e^{-\rho_1 t_1} dt_1 = S e^{-\rho_1 t_2}. \quad (3.133)$$

A second transformation with respect to the variable  $t_2$  leads to the following simple result

$$\overline{m_2^{(x)}}(\rho_1, \rho_2) = \frac{S}{\rho_1 + \rho_2}. \quad (3.134)$$

Let us note that this result could have been obtained also from (3.106), in view of (3.117) and assuming  $\beta \rightarrow \infty$ .

The above considerations can be obviously extended to other relationships of the method of stochastic differential equations.

Let us find the spectral density of uncorrelated input. If the correlation period of the input is much shorter than the period of the frequency at which the spectral density is to be computed, then one can assume  $\cos \omega z = 1$ , in the calculation of integral (3.62) and then:

$$F(\omega) = \frac{2}{\pi} \int_0^{\infty} m_2(z) dz = \frac{S}{\pi}, \quad (3.135)$$

where the quantity  $S$  has the same meaning as before.

In the ideal case when the correlation period equals zero, expression (3.135) is correct at all frequencies from zero to infinity. For a small but finite correlation period it is valid only at sufficiently low frequencies which correspond to periods much longer than the correlation period.

Equation (3.135) shows that a stationary uncorrelated input has a uniform spectrum in the entire frequency range from zero to infinity.

An uncorrelated input has an infinite mean square  $\sigma_{in}^2$ , as follows directly from equation (3.128) when we take there  $t'_1 = t'_2$ . The same result can be obtained from (3.117) by assuming  $\beta \rightarrow \infty$  at  $S = \text{const}$ . Consequently, such an input has infinite energy. This result is the consequence of the assumed idealization of the actual properties of the input.

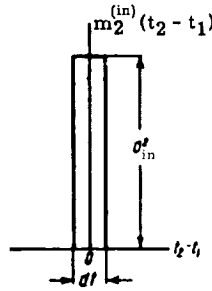


Figure 4. Graph of the second moment of an uncorrelated voltage input

We shall pay attention to the fact that an uncorrelated input can cause a finite response of the system only if it has an infinite mean square, i. e., a finite area  $S$ . To explain this we shall introduce the concept of specific energy of the input, interpreting it as the energy developed in a unit active resistance. If we now consider the input succession of infinitesimal impulses immediately following on each other, the heights of the impulses being considered as uncorrelated, then the graph of the second moment will be of the form shown in Figure 4. and the area  $S$  will represent the mean specific energy of the individual elementary impulses.

Each elementary impulse communicates a certain energy store to the system. Storage of energy takes place in the system only when the impulses applied to it are ordered to some extent, as is the case with determinate as well as with correlated random actions. Then, at infinitely small specific energy of the impulses (finite  $\sigma_{in}^2$ ) their superposition produces a finite effect at the output. In the absence of correlation no energy can be stored in the system and finite response can be elicited only in the case when the specific energy of each elementary impulse is infinite, i. e.,  $\sigma_{in}^2 = \infty$ .

Let us explain the connection between the form of the correlation ratio at the system's output, under uncorrelated input, and the character of the system's specific transient process. For this we shall turn to the RC circuit examined in the preceding section. Let there occur a stationary random process caused by an uncorrelated fluctuating voltage in the circuit whilst the voltage across the capacitor equals  $u_1$  at the moment  $t_1$  (Figure 5). The voltage  $u_2$  across on the capacitor at the moment  $t_2$  can be regarded as the result of the superposition of two processes:

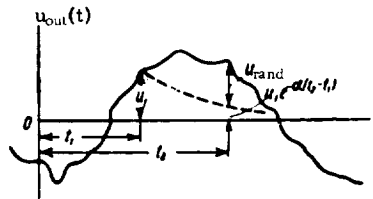


Figure 5. The character of the voltage at the output of an RC circuit under fluctuating input

1) the discharge of the capacitor through the resistance and input source during the time  $t_2 - t_1$  (determinate process), and 2) the simultaneous charging of the capacitor by the input fluctuating voltage (random process).

Thus:

$$u_2 = u_1 e^{-s(t_2 - t_1)} + u_{\text{rand}}. \quad (3.136)$$

Now, multiplying both sides of (3.136) by  $u_1$  and taking the average we obtain:

$$M\{u_1 u_2\} = M\{u_1^2\} e^{-s(t_2 - t_1)} + M\{u_1 u_{\text{rand}}\}. \quad (3.137)$$

The left-hand side of (3.137) represents the second moment of output voltage  $m_2^{(\text{out})}(t_1, t_2)$ . We further have:

$$M\{u_1^2\} = \sigma_{\text{out}}^2 \quad (3.138)$$

and

$$M\{u_1 u_{\text{rand}}\} = 0. \quad (3.139)$$

The latter relation results from the fact that at uncorrelated input there is no correlation between the voltage  $u_{\text{rand}}$  resulting from input fluctuations over the period  $t_2 - t_1$  and the voltage  $u_1$ , caused by these fluctuations at the moment  $t_1$ .

Taking these considerations into account, one writes equation (3.137) in the following form:

$$m_2^{(\text{out})}(t_1, t_2) = \sigma_{\text{out}}^2 e^{-s(t_2 - t_1)}, \quad (3.140)$$

whence we have for the normalized correlation ratio of the output voltage

$$\rho_{\text{out}}(t_1, t_2) = \frac{m_2^{(\text{out})}(t_1, t_2)}{\sigma_{\text{out}}^2} = e^{-s(t_2 - t_1)} \quad (t_2 > t_1), \quad (3.141)$$

i.e., an expression which coincides with (3.115).

It follows from the given results that under an uncorrelated input, the correlation at the output of the system results from some measure of residual response to the earlier applied input. This residual response decreases according to a law which coincides with that of the system's specific transient process.

To conclude this section we would like to remark that in §8 a random function was mentioned, the arbitrarily near-in-time values of which were statistically independent. Such a function is uncorrelated and all the considerations given above are applicable to it. However, if the second moment has the form of (3.128) this is not yet sufficient for inferring that there is no statistical dependence. Such an inference is valid only for a normally distributed function.

## § 17. The Problem of Two RC Circuits with a Common Input

The aim of this and the following section is to give an illustration of the investigation of random processes in linear systems by methods given in this chapter as applied to systems with several inputs and outputs. We shall now examine the following problem. Two RC circuits are given in (Figure 6) with the parameters:

$$\alpha_1 = \frac{1}{R_1 C_1}, \alpha_2 = \frac{1}{R_2 C_2}. \quad (3.142)$$

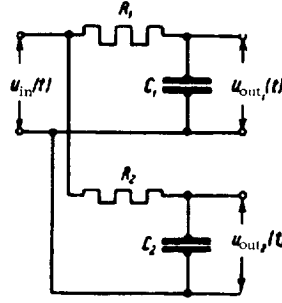


Figure 6. Two RC circuits with a common input

A random voltage  $u_{in}(t)$  of a correlation period much smaller than the time constant of the circuits is applied simultaneously at the input of both circuits. It is required to find an expression for the mixed second moment of the response at both outputs of the system. As in the preceding section, we shall give a solution of the stated problem by using each of the indicated three methods.

The behavior of the examined system is described by a system of two stochastic differential equations of the first order:

$$\frac{du_{out1}(t)}{dt} + \alpha_1 u_{out1}(t) = \alpha_1 u_{in}(t), \quad (3.143)$$

$$\frac{du_{out2}(t)}{dt} + \alpha_2 u_{out2}(t) = \alpha_2 u_{in}(t). \quad (3.144)$$

Let the voltage  $u_{out1}$  be taken at the time  $t_1$  and the voltage  $u_{out2}$  at  $t_2$ . We then have for the sought-for mixed moment the following differential equation

$$\left(\frac{\partial}{\partial t_1} + \alpha_1\right)\left(\frac{\partial}{\partial t_2} + \alpha_2\right)m_2^{(out1,2)}(t_1, t_2) = \alpha_1 \alpha_2 m_2^{(in)}(t_1, t_2). \quad (3.145)$$

Applying formula (3.134), we obtain the double Laplace transform of the second moment of the response.

$$\bar{m}_2^{(out1,2)}(p_1, p_2) = \frac{\alpha_1 \alpha_2 S}{(p_1 + p_2)(p_1 + \alpha_1)(p_2 + \alpha_2)}. \quad (3.146)$$

Returning to the original with respect to the variable  $p_1$  we now obtain:

$$\bar{m}_2^{(out1,2)}(t_1, p_2) = \frac{\alpha_1 \alpha_2 S}{(p_2 + \alpha_2)(p_2 - \alpha_1)} (e^{-\alpha_1 t_1} - e^{-p_2 t_1}). \quad (3.147)$$

Assuming  $t_1 > t_2$  for the sake of definiteness, using the delay theorem of operational calculus, and then returning to the original with respect to the variable  $p_2$  we finally obtain:

$$m_2^{(out1,2)}(t_1, t_2) = \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} S [e^{-\alpha_1(t_1 - t_2)} - e^{-(\alpha_1 t_1 + \alpha_2 t_2)}]. \quad (1.48)$$

By virtue of the full symmetry of both outputs in the case of  $t_2 > t_1$  we can write without further calculation:

$$m_2^{(\text{out } 1, 2)}(t_1, t_2) = \frac{a_1 a_2}{a_1 + a_2} S [e^{-a_1(t_1 - t_2)} - e^{-a_1 t_1 + a_2 t_2}]. \quad (3.149)$$

The obtained results (3.148) and (3.149) are similar in some features to expression (3.109) obtained in the preceding section. We immediately notice they satisfy the initial condition  $m_2^{(\text{out } 1, 2)}(0, 0) = 0$ . Further both expressions have two terms in square brackets, the first term being a function only of the difference  $(t_1 - t_2)$  and corresponding to a stationary random process which approaches the steady-state value for  $t_1 \rightarrow \infty$  and  $t_2 \rightarrow \infty$ . The second term, on the other hand, tends to zero in the same limit.

The nature of the statistical dependence between the responses can be easily traced by writing out the expressions for the normalized correlation ratio of response. From (3.148) and (3.149) in view of (3.125) we have in the steady state, at  $t_1 > t_2$

$$\rho_{\text{out } 1, 2}(t_1, t_2) = \frac{2 \sqrt{a_1 a_2}}{a_1 + a_2} e^{-a_1(t_1 - t_2)}, \quad (3.150)$$

at  $t_2 > t_1$

$$\rho_{\text{out } 1, 2}(t_1, t_2) = \frac{2 \sqrt{a_1 a_2}}{a_1 + a_2} e^{-a_2(t_2 - t_1)}. \quad (3.151)$$

As should have been expected, both expressions yield the same result at  $t_1 = t_2$ :

$$\rho_{\text{out max}} = \frac{2 \sqrt{a_1 a_2}}{a_1 + a_2}. \quad (3.152)$$

The strongest statistical dependence, which becomes functional in the limit, occurs when  $a_1 \rightarrow a_2$  ( $\rho \rightarrow 1$ ). As can be seen from comparison of relations (3.150) and (3.151), the steady-state correlation ratio of both responses is not, in distinction from the autocorrelation ratios of each response, an even function of the difference  $(t_1 - t_2)$ . This may be readily comprehended by considerations analogous to those of the derivations (3.136) - (3.141) of the preceding section.

Let us make the same calculations by the method of impulse characteristics. For brevity we shall restrict ourselves to the case of stationary random processes at the system's outputs.

We shall use expression (3.32) for our calculations, taking in it  $n = 2$ , and, taking into account the uncorrelated nature of the applied input, we shall give it a form analogous to (3.131). Then we shall obtain:

$$m_2^{(\text{out } 1, 2)}(t_1, t_2) = S \int_{-\infty}^{t_2} \xi_1(t_1 - t) \xi_2(t_2 - t) dt, \quad (3.153)$$

where  $t_1 > t_2$ .

By analogy to (3.119), the impulse characteristics of both circuits are expressed as follows:

$$\xi_1(t) = a_1 e^{-a_1 t}; \quad \xi_2(t) = a_2 e^{-a_2 t}. \quad (3.154)$$

Now the computation of the integral of (3.153) gives:

$$\begin{aligned} m_2^{(\text{out } 1, 2)}(t_1, t_2) &= S \int_{-\infty}^{t_2} a_1 e^{-a_1(t_1 - t)} a_2 e^{-a_2(t_2 - t)} dt = \\ &= \frac{a_1 a_2}{a_1 + a_2} S e^{-a_1(t_1 - t_2)}, \end{aligned} \quad (3.155)$$

i. e., a particular case of expression (3.148) for the steady state.

To conclude this section let us consider the calculation of the mixed moment by the spectral method. Finding it amounts to computing the integral of (3.100) wherein, in this case,

$$K_{11}(j\omega) = \frac{a_1}{a_1 + j\omega}, \quad (3.156)$$

$$K_{21}(j\omega) = \frac{a_2}{a_1 + j\omega}, \quad (3.157)$$

$$F_{x_1}(\omega) = \frac{S}{\pi}, \quad F_{x_2}(\omega) = F_{x, x_1}(j\omega) = 0. \quad (3.153)$$

Expression (3.100) assumes now the form

$$\begin{aligned} m_2^{\text{out } 1, 2}(z) &= \frac{1}{2} \int_{-\infty}^{+\infty} K_{11}^*(j\omega) K_{21}(j\omega) F_{x_1}(\omega) e^{j\omega z} d\omega = \\ &= \frac{a_1 a_2}{2} \frac{S}{\pi} \int_{-\infty}^{+\infty} \frac{e^{j\omega z} d\omega}{(a_1 - j\omega)(a_2 + j\omega)} = \frac{a_1 a_2}{a_1 + a_2} S e^{-a_2 z}. \end{aligned} \quad (3.159)$$

However if  $t_1 > t_2$ , then, keeping in mind the remark concerning equation (3.100), we have:

$$\begin{aligned} m_2^{\text{out } 1, 2}(z) &= \frac{1}{2} \int_{-\infty}^{+\infty} K_{11}(j\omega) K_{21}^*(j\omega) F_{x_1}(\omega) e^{j\omega z} d\omega = \\ &= \frac{a_1 a_2}{2} \frac{S}{\pi} \int_{-\infty}^{+\infty} \frac{e^{j\omega z} d\omega}{(a_1 + j\omega)(a_2 - j\omega)} = \frac{a_1 a_2}{a_1 + a_2} S e^{-a_1 z}. \end{aligned} \quad (3.160)$$

Thus, the stated problem has been solved by each of the three methods set forth in this chapter.

#### § 18. The Problem of Two RC Circuits with a Common Output

The electric circuit in this section and the designation of its parameters are shown in Figure 7. The left and right pairs of terminals will be regarded as the two inputs, points a and b as the output. Random voltages of equal mean squares  $\sigma_{in}^2$  are applied across the inputs. Each of these voltages is stationary and uncorrelated, the second one being obtained from the first one by the introduction of a delay time T.

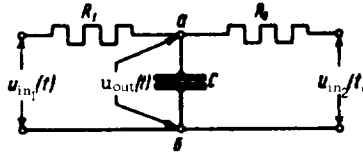


Figure 7. Two RC circuits with a common output

By analogy to (3.128) we can express the second moment of each of the input

voltages as:

$$m_2^{(in)}(t_1, t_2) = S \cdot \delta(t_2 - t_1). \quad (3.161)$$

The mixed moment of both input voltages is determined as follows:

$$m_2^{(12)}(t_1, t_2) = S \cdot \delta(t_2 - t_1 - T), \quad (3.162)$$

where the voltage at the first input is taken at the time  $t_1$  and the voltage at the second one - at  $t_2$ . The objects of investigation are the statistical properties of the output voltage.

We shall first carry out the analysis of the random process by the method of stochastic differential equations. The corresponding differential equation has the form:

$$-\frac{du_{out}}{dt} + (a_1 + a_2)u_{out} = a_1 u_{in_1} + a_2 u_{in_2}, \quad (3.163)$$

where

$$a_1 = \frac{1}{R_1 C}; \quad a_2 = \frac{1}{R_2 C}. \quad (3.164)$$

Equation (3.163) of the given specific problem corresponds to the system of equations (3.22) in the general case. In our case the number of inputs  $m = 2$ , the number of outputs  $n = 1$ , the output reference number assumed the fixed value  $i = 1$ , the input reference number has two possible values  $j = 1, 2$ . Since the system has degenerated into a single equation, one should take  $k = 1$ .

In the differential equation of type (3.24) enter for the second moment of the response the operators:

$$A_{i_1}^{(1,k)} = \frac{\partial}{\partial t_1} + (a_1 + a_2), \quad A_{i_2}^{(1,k)} = \frac{\partial}{\partial t_2} + (a_1 + a_2); \quad (3.165)$$

$$B_{i_1}^{(j,k)} = B_{i_2}^{(j,k)} = B^{(11)} = a_1 \quad (j_1 = j_2 = 1); \quad (3.166)$$

$$B_{i_1}^{(j,k)} = B_{i_2}^{(j,k)} = B^{(21)} = a_2 \quad (j_1 = j_2 = 2). \quad (3.167)$$

In view of expressions (3.165), (3.166) and 3.167) we shall write the mentioned differential equation thus:

$$\begin{aligned} \left[ \frac{\partial}{\partial t_1} + (a_1 + a_2) \right] \left[ \frac{\partial}{\partial t_2} + (a_1 + a_2) \right] m_2^{out}(t_1, t_2) = \\ = (a_1^2 + a_2^2) m_2^{in}(t_1, t_2) + 2a_1 a_2 m_2^{(12)}(t_1, t_2). \end{aligned} \quad (3.168)$$

We shall subject both sides of equation (3.168) to a double Laplace transformation with respect to the variables  $t_1$  and  $t_2$ . We shall take into account expression (3.134) and also the fact that a computation similar to (3.133) yields:

$$\overline{m}_2^{(12)}(p_1, p_2) = \frac{S}{p_1 + p_2} e^{p_1 T}. \quad (3.169)$$

The double transform of the second moment of response can then be expressed as follows:

$$\begin{aligned} \overline{m}_2^{(out)}(p_1, p_2) &= (a_1^2 + a_2^2) \frac{S}{p_1 + p_2} \frac{1}{(p_1 + a_1 + a_2)(p_2 + a_1 + a_2)} + \\ &+ 2a_1 a_2 \frac{S}{p_1 + p_2} e^{p_1 T} \frac{1}{(p_1 + a_1 + a_2)(p_2 + a_1 + a_2)}. \end{aligned} \quad (3.170)$$

Assuming for definiteness  $t_1 > t_2$  and applying the inverse transformation, we obtain for  $t_1 < T$

$$m_2^{(out)}(t_1, t_2) = \frac{S}{2(a_1 + a_2)} (a_1^2 + a_2^2) [e^{-(a_1 + a_2)(t_1 - t_2)} - e^{-(a_1 + a_2)(t_1 + t_2)}], \quad (3.171)$$

for  $t_1 > T$

$$\begin{aligned} m_2^{(out)}(t_1, t_2) &= \frac{S}{2(a_1 + a_2)} [a_1^2 + a_2^2 + 2a_1 a_2 e^{-(a_1 + a_2)T}] \times \\ &\times [e^{-(a_1 + a_2)(t_1 - t_2)} - e^{-(a_1 + a_2)(t_1 + t_2)}]. \end{aligned} \quad (3.172)$$

Let us examine expressions (3.171) and (3.172). We note first that at  $t_1 = t_2 = 0$  we have  $m_2^{(out)}(t_1, t_2) = 0$ . Further, for  $t_1 < T$  the correlation between the input voltages has no effect. One can easily convince oneself that (3.171) is the sum of the moments of output voltages resulting from each of the input voltages separately. The point is that in this case the second input is not yet affected by the fluctuation impulses correlated with the voltage impulses at the first input. At the time  $t_1 = T$  the correlation appears and the second moment undergoes a jump-like increase. The shorter the delay time  $T$ , the higher the relative value of this increase.

If  $t_1 \rightarrow \infty$  and  $t_2 \rightarrow \infty$ , while the time interval  $\tau = |t_2 - t_1|$  remains finite, expression (3.172) assumes the form corresponding to a stationary random process:

$$m_2^{(out)}(\tau) = \frac{S}{2(a_1 + a_2)} [a_1^2 + a_2^2 + 2a_1 a_2 e^{-(a_1 + a_2)T}] e^{-(a_1 + a_2)\tau}. \quad (3.173)$$

If  $T = 0$ , we obtain from (3.173)

$$m_2^{(out)}(\tau) = \frac{1}{2} (a_1 + a_2) S e^{-(a_1 + a_2)\tau}, \quad (3.174)$$

which coincides with the result for one circuit with the parameter  $\alpha = a_1 + a_2$ . In fact, under these conditions we can connect in parallel the two inputs of the circuit, thus uniting the two circuits into one.

We shall carry out the same calculations by the method of impulse characteristics. For making the derivations shorter we shall examine only a stationary random process at the output of the system.

The impulse characteristics of the system are expressed as follows:

$$\xi_{11}(t) = a_1 e^{-(a_1 + a_2)t}, \quad (3.175)$$

$$\xi_{12}(t) = a_2 e^{-(a_1 + a_2)t}. \quad (3.176)$$

The relations (3.175) and (3.176) can be easily obtained from differential equation (3.163), taking one of the inputs as a unit impulse and the other equal to zero.

We shall use expression (3.39) to calculate the second moment of response. Since only a stationary random process is considered, we shall take the lower integration limits of this expression equal to  $-\infty$ . If the moments are of the form (3.161)

and (3.162) one can carry out derivations similar to those preceding formula (3.131) and simplify (3.39) as follows:

$$m_2^{(out)}(t_1, t_2) = S \left[ \int_{-\infty}^{t_1} \xi_{11}(t_1 - t) \xi_{11}(t_2 - t) dt + \right. \\ \left. + \int_{-\infty}^{t_1} \xi_{11}(t_2 - t) \xi_{12}(t_1 + T - t) dt + \int_{-\infty}^{t_1} \xi_{11}(t_1 + T - t) \xi_{12}(t_2 - t) dt + \right. \\ \left. + \int_{-\infty}^{t_1} \xi_{12}(t_1 - t) \xi_{12}(t_2 - t) dt \right]. \quad (3.177)$$

Substituting the values of  $\xi_{11}(t)$  and  $\xi_{12}(t)$  in (3.177) and integrating, we obtain:

$$m_2^{(out)}(t_1, t_2) = \frac{S}{2(a_1 + a_2)} \times \\ \times [a_1^2 + a_2^2 + 2a_1 a_2 e^{-(a_1 + a_2)T}] e^{-(a_1 + a_2)(t_1 - t_2)}, \quad (3.178)$$

i.e., a result which coincides with (3.173).

In conclusion we shall show how to obtain expression (3.178) by the spectral method. In accordance with (3.135) the spectral densities of the applied inputs will be

$$F_{in_1}(\omega) = F_{in_2}(\omega) = \frac{S}{\pi}. \quad (3.179)$$

The mutual spectral density of the inputs in (3.96) is in our case:

$$F_{12}(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} S \cdot \delta(\theta + T) e^{-j\omega\theta} d\theta = \frac{S}{\pi} e^{j\omega T}. \quad (3.180)$$

The transfer ratios of the system are given by

$$K_{11}(j\omega) = \frac{a_1}{a_1 + a_2 + j\omega}. \quad (3.181)$$

$$K_{12}(j\omega) = \frac{a_2}{a_1 + a_2 + j\omega}. \quad (3.182)$$

The application of formula (3.97) gives now the following result:

$$m_2^{(out)}(\tau) = \int_0^{\infty} \frac{a_1^2}{(a_1 + a_2)^2 + \omega^2} \cdot \frac{S}{\pi} \cos \omega \tau d\omega + \\ + \int_0^{\infty} \frac{a_2^2}{(a_1 + a_2)^2 + \omega^2} \cdot \frac{S}{\pi} \cos \omega \tau d\omega + \\ + \frac{1}{2} \int_{-\infty}^{+\infty} \left[ \frac{a_1}{a_1 + a_2 + j\omega} \cdot \frac{a_2}{a_1 + a_2 - j\omega} + \frac{a_1}{a_1 + a_2 - j\omega} \cdot \frac{a_2}{a_1 + a_2 + j\omega} \right] \times \\ \times \frac{S}{\pi} e^{j\omega T} e^{j\omega \tau} d\omega = \frac{S}{2(a_1 + a_2)} [a_1^2 + a_2^2 + 2a_1 a_2 e^{-(a_1 + a_2)T}] \times \\ \times e^{-(a_1 + a_2)\tau}, \quad (3.183)$$

which is identical with equation (3.173).

## Chapter Four

### SOME LINEAR PROBLEMS IN THE THEORY OF RANDOM PROCESSES

#### §19. One-Dimensional Brownian Motion

In the absence of an external force field the one-dimensional Brownian motion of a particle is described, in accordance with the second law of Newton, by the following differential equation

$$m \frac{dv}{dt} + rv = f(t), \quad (4.1)$$

where  $m$  and  $v$  are, respectively, the mass and the velocity of the particle,  $r$  is the coefficient of friction encountered by the particle,  $f(t)$  is the projection of the total force acting on the particle, as a result of molecular motion, on the direction along which the motion of the particle is considered.

Equation (4.1) can also be expressed differently:

$$\frac{dv}{dt} + \alpha v = g(t), \quad (4.2)$$

where the coefficient  $\alpha$  is defined by Stokes's law:

$$\alpha = \frac{6\pi a \eta}{m}. \quad (4.3)$$

Here  $a$  is the radius of the particle and  $\eta$  is the viscosity coefficient of the surrounding fluid.

Equation (4.2) is called the Langevin equation.

Since the function  $g(t)$  varies much more rapidly than the particle velocity, this function can be considered as uncorrelated.

Equation (4.2) is entirely analogous to the equation (3.102), which describes a random excitation of an RC circuit. Consequently, the already obtained result (3.109) can be used. In this connection one ought to take into account that because of the noncorrelation of the input it must be assumed that  $\beta \rightarrow \infty$ , and to note equation (3.124), as well as the fact that the factor  $\alpha$  of the equation (3.102) is missing in the right-hand side of (4.2). Having made these allowances, the second moment of the particle velocity can be written as follows: for  $t_1 > t_2$

$$m_2^{(v)}(t_1, t_2) = \frac{S}{2\alpha} [e^{-\alpha(t_1-t_2)} - e^{-\alpha(t_1+t_2)}] = \frac{S}{\alpha} e^{-\alpha t_1} \operatorname{sh} \alpha t_2, \quad (4.4)$$

for  $t_2 > t_1$

$$m_2^{(v)}(t_1, t_2) = \frac{S}{2\alpha} [e^{-\alpha(t_2-t_1)} - e^{-\alpha(t_1+t_2)}] = \frac{S}{\alpha} e^{-\alpha t_2} \operatorname{sh} \alpha t_1. \quad (4.5)$$

Under steady-state fluctuation conditions the mean square velocity of the particle will be

$$\sigma_v^2 = \frac{S}{2a}. \quad (4.6)$$

by analogy with (3.125).

For the determination of the quantity  $S$  we can use the proposition of statistical physics, which states that a system in a state of stationary thermal motion has a mean energy of  $kT/2$  for each degree of freedom [equipartition theorem]. In our case

$$\frac{m\sigma_v^2}{2} = \frac{1}{2} kT. \quad (4.7)$$

This relation in conjunction with (4.6) gives

$$S = 2a\sigma_v^2 = 2a \frac{kT}{m}. \quad (4.8)$$

To avoid misunderstandings, let us recall that the function  $g(t)$  of (4.2) is equal to

$$g(t) = \frac{f(t)}{m}. \quad (4.9)$$

Therefore, the quantity  $S$ , which characterizes the intensity of molecular agitation, depends not only on the parameters of the liquid but also on the mass of the particle.

If at the initial moment  $t=0$  the particle was at the origin of the coordinates, ( $x=0$ ), then at the time  $t$  its position will be

$$x(t) = \int_0^t v(t) dt. \quad (4.10)$$

Expression (4.10) is an integral transformation of the random function  $v(t)$ . This permits to obtain the mean square of the coordinate at the moment  $t$  in the following way

$$\sigma_x^2 = \int_0^t \int_0^t m_2^{(v)}(t_1, t_2) dt_1 dt_2. \quad (4.11)$$

Let us calculate the inner integral of (4.11):

$$\begin{aligned} \int_0^t m_2^{(v)}(t_1, t_2) dt_1 &= \int_0^{t_1} \frac{S}{a} e^{-at_1} \operatorname{sh} at_1 dt_1 + \int_{t_1}^t \frac{S}{a} e^{-at_1} \operatorname{sh} at_1 dt_1 = \\ &= \frac{S}{a^2} \{1 - e^{-at_1} - e^{at_1} \operatorname{sh} at_1\}. \end{aligned} \quad (4.12)$$

Now the calculation of the outer integral of (4.11) gives the following result:

$$\sigma_x^2 = \frac{S}{2a^3} \{2at - 3 + 4e^{-at} - e^{-2at}\}. \quad (4.13)$$

The obtained expression shows that the coordinate  $x(t)$  of the particle is essentially a nonstationary random function. At  $t=0$  we have, as expected,  $\sigma_x^2 = 0$ . With increasing  $t$  the mean square  $\sigma_x^2$  grows indefinitely. This is explained by the absence of a returning force.

If the time  $t$  is sufficiently long ( $at \gg 1$ ), then, in the braces of (4.13) all terms except the first can be neglected, and the formula (4.13) reduces to

$$\sigma_x^2 = \frac{S}{a^2} t. \quad (4.14)$$

With the use of previously obtained expressions (4.3) and (4.8) for the quantities  $\alpha$  and  $S$  we can finally write

$$\sigma_x^2 = \frac{kT}{3\pi a \eta} t. \quad (4.15)$$

This result was first obtained by other methods, by A. Einstein. The experimental check by J. Perrin gave satisfactory agreement.

## § 20. Thermal Noise in Electric Circuits

It is known that a random thermal motion in the conductors of any electric circuit gives rise to fluctuating currents and voltages, which are often called noise. For calculation of the correlation functions of these fluctuations, any of the methods considered in the previous chapter can be used. But any one of them must be supplemented by a method for calculating the intensity of the fluctuations.

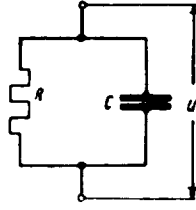


Figure 8. Elementary RC circuit

Let us consider the intensity of electrical fluctuations in an elementary RC circuit (Figure 8). We shall assume equipartition, as in the foregoing section, i. e., that in any system which is in thermal motion the mean energy of fluctuation is equal to  $kT/2$  for each degree of freedom. In our case the state of the system is completely characterized by a single coordinate: the voltage  $u$  across the circuit. We have, therefore:

$$\frac{1}{2} C \sigma_u^2 = \frac{1}{2} kT, \quad (4.16)$$

where  $\sigma_u^2$  is the mean square of the voltage across the circuit.

From (4.6) we obtain:

$$\sigma_u^2 = \frac{kT}{C}. \quad (4.17)$$

The fluctuating voltage  $u$  can be considered as originating from an equivalent generator with a random electromotive force  $e(t)$ , which has been connected in series with the circuit (Figure 9). Since the thermal motion in the conductors is extremely rapid the electromotive force  $e(t)$  is considered as uncorrelated.

For an uncorrelated random excitation of an RC circuit we obtained in § 15 the equation (3.118), which we shall write here in the following form:

$$\sigma_u^2 = \frac{1}{2RC} S. \quad (4.18)$$

From (4.17) and (4.18) we obtain

$$S = 2kTR. \quad (4.19)$$

The spectral density of the electromotive force  $e(t)$  is, in accordance with (3.135), equal to

$$F(\omega) = \frac{2}{\pi} kTR. \quad (4.20)$$

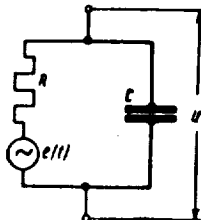


Figure 9. Simulation of electrical fluctuations in an RC circuit by introduction of an equivalent uncorrelated electromotive force

Formula (4.20) for the spectral density of a fluctuating electromotive force was first obtained by H. Nyquist [12]. The substance of Nyquist's considerations is also expounded in the book of S. Goldman [13]. The analysis given by these two authors is considerably more complicated than ours. But it is more rigorous since, unlike us, they do not make the a priori assumption of uncorrelatedness of the random electromotive force.

Let us clarify which of the elements of the RC circuit (Figure 8) is the source of fluctuations. We initially assume that the fluctuations are generated in the resistance as well as in the capacitor. Then it can be said that the random electromotive force  $e(t)$  (Figure 9) can be considered as the sum of two electromotive forces  $e_R(t)$  and  $e_C(t)$ , of which the first corresponds to fluctuations generated in the resistance, and the second to the fluctuations originating in the capacitor (Figure 10).

As the capacitor is a reactance, the power produced in it by the electromotive force of the resistance is zero, i. e., the resistance does not transmit its thermal motion to the capacitor. Therefore, the assumption of a nonvanishing fluctuation electromotive force  $e_C(t)$  in the capacitor, leads to the consequence that the resistance is continuously heated by fluctuation currents generated in the capacitor, i. e., its temperature must increase beyond all limits. This consequence contradicts the law of energy conservation. The circumstance that through the resistance  $R$  flows its own noise current does not lead to a change in its temperature, as the energy of this current is taken from the energy of thermal motion and is transformed back into heat.

Thus, the only source of the fluctuations is the resistance. The same considerations could be repeated after replacing the capacitance by an inductance. Thus, reactive components do not produce noise. These results can be applied to any resistances and reactances in a composite electric circuit.

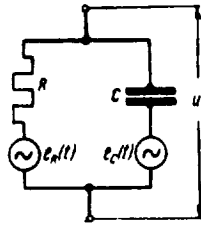


Figure 10. Resolution of the equivalent fluctuating electromotive force into two components

## § 21. Thermal Noise in an Electric Oscillation Circuit

The above results can be used for calculating the thermal noise voltage in an electric oscillation circuit (Figure 11). According to what has been said in the foregoing section, the only source of the fluctuations in the circuit is the resistance  $r$ . The "noisy" resistance  $r$  can be represented as a "noiseless" resistance of the same value in series with an uncorrelated noise electromotive force of spectral density

$$F(\omega) = \frac{2}{\pi} kTr. \quad (4.21)$$

Then the equivalent circuit of Figure 12 can be used for the calculation of the noise voltage  $u$ .

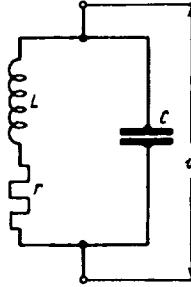


Figure 11. Electric oscillation circuit

The transfer ratio of the noise electromotive force is

$$K(j\omega) = \frac{1/j\omega C}{r + j\omega L + 1/j\omega C} = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + j2\alpha\omega}, \quad (4.22)$$

where  $\omega_0 = 1/\sqrt{LC}$  is the natural frequency of the circuit and  $\alpha = r/2L$  the damping factor

The square of the modulus of the transfer ratio will be:

$$|K(j\omega)|^2 = \frac{\omega_0^4}{(\omega^2 - \omega_0^2)^2 + 4\alpha^2\omega^2}. \quad (4.23)$$

The spectral density of the noise voltage  $u$  in the circuit is determined from (4.21) and (4.23) in the following way:

$$F_u(\omega) = \frac{2kT}{\pi} r \frac{\omega_0^4}{(\omega^2 - \omega_0^2)^2 + 4\alpha^2\omega^2}. \quad (4.24)$$

Let us examine the structure of this equation. First, we shall determine the effective component of the equivalent resistance of the circuit (Figure 11):

$$R(\omega) = \operatorname{Re} \left[ \frac{(r + j\omega L) 1/j\omega C}{r + j\omega L + 1/j\omega C} \right] = r \frac{\omega_c^2}{(\omega^2 - \omega_0^2)^2 + 4\alpha^2\omega^2}. \quad (4.25)$$

By combining the equations (4.24) and (4.25) we obtain for the first of them the following simple form:

$$F_u(\omega) = \frac{2}{\pi} kT \cdot R(\omega). \quad (4.26)$$

A comparison of (4.26) with (4.20) shows that the expression (4.20) can be used in the calculation of the noise electromotive force as the effective resistance as well as the effective component of the complex resistance.

If the Q-factor ( $Q = \omega_0 L/r$ ) is sufficiently high, the effective component  $R(\omega)$  of the equivalent resistance of the circuit and the spectral density  $F_u(\omega)$  of the noise voltage, which is proportional to  $R(\omega)$ , have a sharp maximum at a frequency practically equal to the natural frequency  $\omega_0$  of the circuit. This maximum is the sharper the higher the Q-factor. Thus, with a high Q, the main part of the fluctuation energy is concentrated in a narrow frequency band, centered about the natural frequency  $\omega_0$  of the circuit. This means, that under the indicated conditions the fluctuations resemble harmonic oscillations having the natural frequency of the circuit.

Using expressions (3.63) and (4.24) let us compute the second moment of the noise voltage across the circuit:

$$\begin{aligned} m_2^{(u)}(\tau) &= \int_0^\infty F_u(\omega) \cos \omega \tau d\omega = \\ &= \frac{2kTr\omega_0^4}{\pi} \int_0^\infty \frac{\cos \omega \tau d\omega}{(\omega^2 - \omega_0^2)^2 + 4\alpha^2\omega^2} = \\ &= \frac{kT}{C} e^{-\omega_1 \tau} \left( \cos \omega_1 \tau + \frac{\alpha}{\omega_1} \sin \omega_1 \tau \right), \end{aligned} \quad (4.27)$$

where

$$\omega_1 = \sqrt{\omega_0^2 - \alpha^2}.$$

The mean square of this voltage can be found by assuming  $\tau = 0$  in (4.27)

$$\sigma_u^2 = \frac{kT}{C}. \quad (4.28)$$

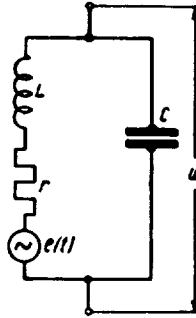


Figure 12. Simulation of electrical fluctuations in an oscillation circuit by an equivalent uncorrelated electromotive force

If only the last result were required, it could have been obtained immediately from the relation

$$\frac{1}{2} C \sigma_u^2 = \frac{1}{2} kT. \quad (4.29)$$

which is known from the foregoing sections.

Let us continue the examination of equation (4.27). Noting that the quantities  $x$  and  $\omega_0$  are connected by the obvious relation

$$\alpha = \frac{1}{2} \frac{\omega_0}{Q} \quad (4.30)$$

and assuming that the Q-factor of the circuit is sufficiently high (generally  $Q \gg 100$ ), (4.27) can be simplified to

$$m_2^{(u)}(\tau) = \sigma_u^2 e^{-\alpha \tau} \cos \omega_0 \tau. \quad (4.31)$$

In order to grasp the meaning of this equation we shall consider the harmonic oscillation

$$x(t) = X \cos(\omega_0 t + \varphi), \quad (4.32)$$

where  $\varphi$  is the random initial phase, all values of which are equally probable. Under this assumption the second moment of the oscillation can be calculated as follows:

$$\begin{aligned} m_2^{(x)}(t_1, t_2) &= M[x(t_1) \cdot x(t_2)] = \\ &= M[X^2 \cos(\omega_0 t_1 + \varphi) \cdot \cos(\omega_0 t_2 + \varphi)] = \\ &= \frac{1}{2} X^2 \{M[\cos \omega_0(t_2 - t_1)] + M[\cos \{\omega_0(t_1 + t_2) + 2\varphi\}]\} \end{aligned} \quad (4.33)$$

and finally

$$m_2^{(x)}(t_1, t_2) = m_2^{(x)}(\tau) = \frac{1}{2} X^2 \cos \omega_0 \tau = \sigma_x^2 \cos \omega_0 \tau, \quad (4.34)$$

where

$$\tau = |t_2 - t_1|; \quad \sigma_x^2 = \frac{1}{2} X^2.$$

As pointed out above, the voltage fluctuations in a circuit with a high Q-factor are nearly harmonic oscillations. They contain, however, a certain random element. The larger the random element the weaker is, at fixed  $\tau$ , the statistical dependence and the smaller its quantitative measure—the second moment  $m_2^{(x)}(\tau)$ . This randomness of the oscillatory process can be accounted for by introducing into the equation (4.34) the factor  $\psi(\tau)$ , which is equal to unity at  $\tau = 0$ , and decreases with increasing  $\tau$ . Thus we obtain

$$m_2^{(x)}(\tau) = \sigma_x^2 \psi(\tau) \cos \omega_0 \tau. \quad (4.35)$$

Equation (4.35) is typical of those stationary random process which closely resemble harmonic oscillations. In particular, of such a form is the second moment of the fluctuations at the output of any selective system, i.e., a system with sufficiently sharp resonance properties, which is subjected to an uncorrelated input. The form of the function  $\psi(\tau)$  is determined by the structure of the system.

For a single oscillation circuit we obtain from (4.31):

$$\psi(\tau) = e^{-\alpha \tau}. \quad (4.36)$$

It follows from previous considerations that the damping factor  $\alpha$  entering (4.36) characterizes the degree of randomness of the fluctuations.

The character of voltage fluctuations in an oscillation circuit or at the output of any selective system can be easily found from purely qualitative considerations, by following the method of impulse characteristics. As already remarked in § 13, an uncorrelated input can be considered as a succession of impulse functions with uncorrelated areas, following immediately upon each other. A single needle-shaped input pulse imparts to the circuit a certain store of energy, initiating damped oscillations of frequency  $\omega_0$ . The resulting process is the sum of infinitely many elementary oscillations of this kind. The oscillations from previous pulses are attenuated, but the energy of the fluctuations is replenished by newly produced oscillations. The sum of any number of oscillations of same frequency  $\omega_0$  is also an oscillation of the same frequency  $\omega_0$ .

Since the initial amplitudes of the continually arising elementary oscillation processes change at random from one process to another, the amplitude and the phase of the resulting oscillation are also continually fluctuating, i.e., they are random functions of time.

If the statistical properties of the instantaneous noise voltage are known, the statistical properties of the random amplitudes and phases of the fluctuation process can be investigated. But this necessitates the use of nonlinear transformations of random functions, which we have not yet considered. We postpone, therefore, the further consideration of the above problem to the sixth chapter.

## § 22. Thermal Motion of a Galvanometer

When working with highly sensitive galvanometers one has to take into account that the moving system of the instrument is in a state of incessant agitation, analogous to the Brownian motion, which renders very difficult the measurement of very small currents. This motion is caused, on the one hand, by the molecular motion of the air surrounding the system, and on the other hand by the electric fluctuation currents in the galvanometer coil.

We shall consider first the random fluctuations of the moving system of the instrument, with the circuit of its coil open. Then no current flows through the coil and the behavior of the galvanometer can be described by the usual differential equation of torsional oscillations

$$I \frac{d^2\theta}{dt^2} + r \frac{d\theta}{dt} + D\theta = M(t), \quad (4.37)$$

where  $\theta$  is the deflection angle of the moving system of the instrument,  $I$  - its moment of inertia,  $r$  - the coefficient of friction,  $D$  - the rigidity of suspension, and  $M(t)$  - the random torque exerted on the moving system by the molecular motion of the surrounding medium. It is natural to consider the random function  $M(t)$  as uncorrelated. We shall examine only stationary fluctuation conditions, using the method of impulse characteristics.

Assuming the right-hand side of equation (4.37) to be a unit impulse function and applying to both sides of this equation the Laplace transformation, we obtain the following expression, representing the impulse characteristic of the galvanometer:

$$\bar{\theta}(p) = \frac{1}{p^2 I + pr + D} \quad (4.38)$$

and by introducing  $\alpha = r/2I$  - the coefficient of damping,  $\omega_0 = \sqrt{D/I}$  - the natural frequency of the moving system,  $\beta^2 = \alpha^2 - \omega_0^2$ , we obtain

$$\bar{\theta}(p) = \frac{1}{I[(p + \alpha)^2 - \beta^2]}. \quad (4.39)$$

In the following, only the aperiodic case of the motion will be considered, when  $\alpha^2 > \omega_0^2$ , i.e.,  $\beta^2 > 0$ . Then the inverse transformation of (4.39) gives

$$\theta(t) = \frac{1}{j\beta} e^{-\alpha t} \text{sh } \beta t. \quad (4.40)$$

For the calculation of the second moment of the angle  $\theta$  we use equation (3.132). Substituting in its right-hand side the expression (4.40) for the impulse characteristic of the instrument and integrating, we obtain

$$m_2^{(\theta)}(\tau) = \frac{S}{2rD} e^{-\alpha|\tau|} (\text{ch } \beta\tau + \frac{\alpha}{\beta} \text{sh } \beta|\tau|). \quad (4.41)$$

Assuming in (4.41)  $\tau=0$ , we obtain the mean square of the angle  $\theta$ :

$$\sigma_\theta^2 = \frac{S}{2rD}. \quad (4.42)$$

To find the value of the unknown quantity  $S$  we shall apply the same method as in foregoing sections. The component of the mean fluctuation energy with respect to the coordinate  $\theta$  is

$$\frac{1}{2} D\sigma_\theta^2 = \frac{1}{2} kT. \quad (4.43)$$

Combining (4.42) and (4.43) we get

$$S = 2kTr. \quad (4.44)$$

The obtained result calls for two comments. First, its similarity to the formula (4.19), which was obtained in § 20 for the noise electromotive force of the resistance, should be noted. Second, we note the double role played by the medium which surrounds the moving system of the instrument. On the one hand the medium opposes the motion with a moment proportional to the coefficient of friction  $r$ , on the other hand it causes the random torque  $M(t)$ , whose value  $S$  is proportional to the same coefficient  $r$ .

In view of equation (4.44) we can put relation (4.41) into its final form:

$$m_2^{(\theta)}(\tau) = \frac{kT}{D} e^{-\alpha|\tau|} (\text{ch } \beta\tau + \frac{\alpha}{\beta} \text{sh } \beta|\tau|). \quad (4.45)$$

As a check on this result let us calculate the component of the mean energy with respect to the other independent coordinate of the moving system - the angular rotation velocity  $\omega = d\theta/dt$ . As was shown in § 7, on differentiating a random function, its second moment is differentiated twice. If the differentiated function is stationary, as in the present case, this differentiation is done with respect to the time interval  $\tau$  and the obtained value is taken with the opposite sign (formula (2.41)). Thus:

$$m_2^{(\omega)}(\tau) = - \frac{d^2 m_2^{(\theta)}(\tau)}{d\tau^2}. \quad (4.46)$$

Differentiating twice and then putting  $\tau=0$  we obtain:

$$\sigma_\omega^2 = \frac{kT}{I}. \quad (4.47)$$

The mean energy of fluctuation along the coordinate  $\omega$  is, as expected,

$$\frac{1}{2} I\sigma_\omega^2 = \frac{1}{2} kT. \quad (4.48)$$

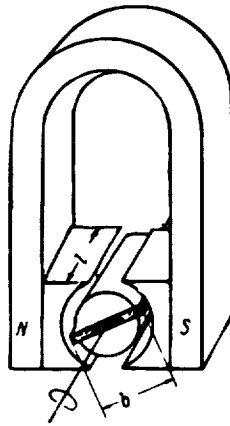


Figure 13. Moving system of a galvanometer in a magnetic field

So far we assumed that the circuit of the galvanometer coil is open. Now let us suppose this circuit to be closed by the resistance  $R$ . For simplicity we shall assume  $R$  to be so large, that the resistance of the coil and the action of its self-induced electromotive force can be neglected. First we shall examine the motion of the moving system of the instrument as caused by the action of the surrounding medium. In that case the closing of the circuit should be considered only as equivalent to the appearance of an additional retarding moment, caused by the current induced in the coil.

Let us calculate the electromotive force induced in the coil by the rotation of the moving system. For this we shall use Figure 13. The field in the gap of the magnetic circuit of the instrument can be considered as radial, in a first approximation. In addition, it can be assumed that on rotation of the coil, its sides parallel to the axes move in a uniform magnetic field. Let  $v$  be the velocity of this movement,  $n$  the number of windings of the coil, and  $B$  the magnetic induction in the gap (other designations are shown in the figure). Then the induced electromotive force is

$$e = 2Blvn. \quad (4.49)$$

Since the velocity  $v$  is expressed by the angular velocity of rotation as follows:

$$v = \omega \frac{b}{2} = \frac{b}{2} \frac{d\theta}{dt}, \quad (4.50)$$

formula (4.49) can be given its final form:

$$e = A \frac{d\theta}{dt}, \quad (4.51)$$

where

$$A = Blbn. \quad (4.52)$$

Under the above assumptions this electromotive force generates in the circuit of the instrument the current

$$i = \frac{e}{R} = \frac{A}{R} \frac{d\theta}{dt}. \quad (4.53)$$

Interacting with the magnetic field, the current  $i$  generates the retarding torque

$$M_{\text{ret}} = B i l n b = A i = \frac{A^2}{R} \frac{d\theta}{dt}. \quad (4.54)$$

By introducing into the left-hand side of differential equation (4.37) the additional retarding torque (4.54), we obtain:

$$I \frac{d^2\theta}{dt^2} + r' \frac{d\theta}{dt} + D\theta = M(t). \quad (4.55)$$

Here

$$r' = r + \frac{A^2}{R}. \quad (4.56)$$

The statistical characteristics of the right side of (4.37), as expressed by the formula (4.44), remains unchanged. Therefore, without further calculation, we can use the ready result (4.41), by replacing it in  $r$  by  $r'$  and retaining the previous value of  $S$ , as expressed by formula (4.44). Thus

$$m_2^{(0)}(\tau) = \frac{kT}{D} \frac{r}{r'} e^{-\alpha'|\tau|} (\text{ch } \beta'\tau + \frac{\alpha'}{\beta'} \text{sh } \beta'\tau | \tau |), \quad (4.57)$$

where

$$\alpha' = \frac{r'}{2I}; \quad \beta' = \sqrt{\alpha'^2 - \omega_0^2}. \quad (4.58)$$

For the mean square of the fluctuations of the angle  $\theta$ , which are produced by the molecular motion of the medium surrounding the galvanometer, we obtain:

$$\sigma_{\theta}^2 = \frac{kT}{D} \frac{r}{r'}. \quad (4.59)$$

The mean square of the fluctuations due to thermal motion of the medium is thus seen to decrease owing to the retarding torque of the forces on the induced current.

Let us now examine the random oscillations of the angle  $\theta$ , which are produced by the electromotive force  $e_1(t)$ , generated in the resistance  $R$ . The random torque, generated by the action of this electromotive force, is equal to

$$M_1(t) = A i_1(t) = \frac{A}{R} e_1(t). \quad (4.60)$$

In accordance with § 20 this electromotive force can be considered as uncorrelated, and its magnitude  $S_{e_1}$  is expressed by formula (4.19). Therefore, the torque  $M_1(t)$  must be also considered as uncorrelated, and its magnitude  $S_{M_1}$  is determined by:

$$S_{M_1} = \frac{A^2}{R^2} S_{e_1} = 2kT \frac{A^2}{R}. \quad (4.61)$$

The random function  $\theta_2$  satisfies the differential equation (4.55), in the right-hand side of which the torque  $M_1(t)$  replaces the torque  $M(t)$ . Therefore, the expression (4.41) in which  $r'$  replaces  $r$  and the quantity  $S$  is defined by formula (4.61) is valid for the second moment of the random angle  $\theta_2$ . Thus,

$$m_2^{(0)}(\tau) = \frac{kTA^2}{Dr'R} e^{-\alpha'|\tau|} (\text{ch } \beta'\tau + \frac{\alpha'}{\beta'} \text{sh } \beta'\tau | \tau |). \quad (4.62)$$

The mean square of the fluctuations of the angle  $\theta_2$  is equal to

$$\sigma_{\theta_2}^2 = \frac{kTA^2}{Dr'R}. \quad (4.63)$$

The random angles  $\theta_1$  and  $\theta_2$  are statistically independent of each other. Therefore the mean square of the resulting angle  $\theta = \theta_1 + \theta_2$  is obtained from (4.56), (4.59) and (4.63) as follows

$$\sigma_{\theta}^2 = \sigma_{\theta_1}^2 + \sigma_{\theta_2}^2 = \frac{kTr}{Dr'} + \frac{kTA^2}{Dr'R} = \frac{kT}{Dr'} \left( r + \frac{A^2}{R} \right) = \frac{kT}{D}, \quad (4.64)$$

i.e., the mean energy of the fluctuations along the coordinate  $\theta$  is, as before, equal to

$$\frac{D\sigma_{\theta}^2}{2} = \frac{1}{2} \frac{kT}{D} D = \frac{1}{2} kT. \quad (4.65)$$

Thus, closing of circuit of the galvanometer has no influence on the mean square of the fluctuations of the angle  $\theta$ . In other words, whatever the mechanism of the transfer of thermal motion to the system, the mean square of its fluctuations along each of the independent coordinates remains unchanged.

From this constancy of the mean square of fluctuations in no way follows that the character of the random motion of the moving system of the instrument is independent of the quantity  $\alpha'$ . This can be easily seen by considering the spectral density of fluctuations. Their second moment can be represented as follows, in accordance with the results obtained above:

$$m_2^{(h)}(\tau) = \sigma_{\theta}^2 e^{-\alpha'|\tau|} \left( \cosh \beta' \tau + \frac{\alpha'}{\beta'} \sinh \beta' |\tau| \right). \quad (4.66)$$

By substituting this expression in formula (3.62) and integrating, we obtain after some simple transformations:

$$F_{\theta}(\omega) = \frac{4}{\pi} \alpha' \omega_0^2 \sigma_{\theta}^2 \frac{1}{(\omega^2 - \omega_0^2)^2 + 4\alpha'^2 \omega^2}. \quad (4.67)$$

Introducing the dimensionless frequency  $\xi = \omega/\omega_0$  and the damping factor  $d = 2\alpha'/\omega_0$  we obtain finally:

$$F_{\theta}(\omega) = \frac{2}{\pi} \frac{d}{\omega_0} \sigma_{\theta}^2 \varphi(\xi, d), \quad (4.68)$$

where

$$\varphi(\xi, d) = \frac{1}{(\xi^2 - 1)^2 + d^2 \xi^2}. \quad (4.69)$$

The function  $\varphi(\xi, d)$  characterizes the dependence of the spectral density of fluctuations on the frequency. The graph of this dependence is shown in Figure 14. It can be seen from it that at sufficiently small damping factors  $d$  (galvanometer circuit open, friction coefficient  $r$  small) the spectral density of fluctuations has a sharp maximum in the range of frequencies near the natural frequency of the moving system. In other words, the character of fluctuations is similar to the character of harmonic oscillations of frequency  $\omega_0$ . On the other hand, for large values of the damping factor, the fluctuations have a highly random character.

The thermal motion of measuring instruments and its influence on the accuracy of measurement are considered in detail in the monograph by V.L. Granovskii [14].

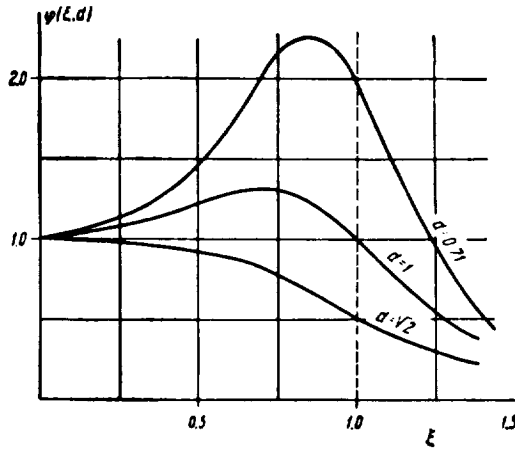


Figure 14. Spectral density of galvanometer fluctuations as a function of frequency

### § 23. The Passage of Irregular Telegraph Signals through a Linear Filter

Let us consider a telegraph signal consisting of a sequence of elements of two kinds (so called binary transmission). The simplest element of the signal is a rectangular pulse of amplitude  $A$  and duration  $T$ . We assume that all pulses follow directly upon each other, have equal characteristic parameters  $A$  and  $T$ , but can have with equal likelihood positive or negative sign. Moreover, we assume that the signs of individual pulses are statistically independent of each other. This is the simplest mathematical model of a telegraph signal, which we shall adopt in the following considerations.

Let us find the second moment of such a signal  $a(t)$ . At  $\tau > T$ , by virtue of the statistical independence of the signs of two neighboring, non-simultaneous pulses, we have:

$$m_2^{(a)}(\tau) = M[a(t) \cdot a(t + \tau)] = 0. \quad (4.70)$$

Let now  $\tau \leq T$ . We choose a certain time moment  $t$ , contained by one of the pulses. Let us assume that the time interval between the beginning of this pulse and the moment  $t$  is a random variable, which can have with equal probability any value between zero and  $T$ . A later moment  $t + \tau$  can fall within the same pulse or the next one. The first case takes place if  $0 < t < T - \tau$ . We have

$$M[a(t) \cdot a(t + \tau)] = A^2. \quad (4.71)$$

The probability of the first case is equal to  $\frac{T - \tau}{T} = 1 - \frac{\tau}{T}$ . Hence if  $T - \tau < t < T$ , the time moment  $t + \tau$  is contained by the next pulse. Then

$$M[a(t) a(t + \tau)] = 0. \quad (4.72)$$

The probability of the second case is  $\tau/T$ . Averaging the product  $a(t) \cdot a(t + \tau)$  over all possible initial positions of the pulse, we have:

$$m_2^{(a)}(\tau) = A^2 \left(1 - \frac{\tau}{T}\right) + 0 \cdot \frac{\tau}{T} = A^2 \left(1 - \frac{\tau}{T}\right). \quad (4.73)$$

Since all the pulses are completely equivalent, this result is valid for the inclusion of the time moment  $t$  within any of the pulses. Thus, (4.73) is the

sought-for expression for the second moment of the telegraph signal. The mean square of the signal ordinate is  $\sigma^2 = A^2$ . Therefore, formula (4.73) can be written as follows:

$$m_2^{(a)}(\tau) = \sigma^2 \left(1 - \frac{\tau}{T}\right). \quad (4.74)$$

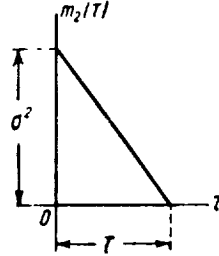


Figure 15. Graph of the second moment of an irregular telegraph signal

The graph corresponding to equation (4.74) is shown in Figure 15. Using the formula (3.62) we find the spectral density of the signal

$$\begin{aligned} F(\omega) &= \frac{2}{\pi} \int_0^T \sigma^2 \left(1 - \frac{\tau}{T}\right) \cos \omega \tau d\tau = \\ &= \frac{2}{\pi} \sigma^2 \frac{1 - \cos \omega T}{\omega^2 T}. \end{aligned} \quad (4.75)$$

The spectrum of the signal is shown in Figure 16. We have now obtained the necessary statistical characteristics of the signal and can proceed to the analysis of its action on a linear filter.

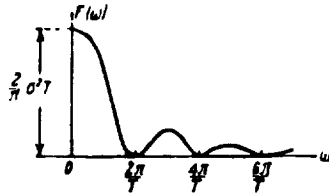


Figure 16. Graph of the spectral density of an irregular telegraph signal

Let us consider an RC circuit (Figure 2) as the simplest filter. Its behavior is described by the differential equation (3.102).

The filter causes distortions of the signal, i. e., there appears a certain error

$$\varepsilon(t) = u_{\text{out}}(t) - u_{\text{in}}(t), \quad (4.76)$$

which is, as the two terms of the right-hand side of (4.76),  $\varepsilon$  random time-function. We can use the mean square  $\sigma_\varepsilon^2$  of the error  $\varepsilon(t)$  as a measure of the distortion.

Let us write the differential equation (3.102) as follows:

$$\frac{d(u_{\text{out}} - u_{\text{in}})}{dt} + a(u_{\text{out}} - u_{\text{in}}) = -\frac{du_{\text{in}}}{dt} \quad (4.77)$$

or, differently,

$$\frac{d\varepsilon}{dt} + a\varepsilon = -\frac{du_{\text{in}}}{dt}. \quad (4.78)$$

We shall use the spectral method. Then we have to calculate the frequency characteristic of the filter error. For this we assume

$$u_{in} = \dot{U}_{in} e^{j\omega t}, \quad e = \dot{E} e^{j\omega t} \quad (4.79)$$

and substitute these values into the equation. We get:

$$j\omega \dot{E} + a\dot{E} = -j\omega \dot{U}_{in}, \quad (4.80)$$

whence

$$K(j\omega) = \frac{\dot{E}}{\dot{U}_{in}} = -\frac{j\omega}{a + j\omega}. \quad (4.81)$$

The square of the modulus of the frequency characteristic of the error is equal to

$$|K(j\omega)|^2 = \frac{\omega^2}{a^2 + \omega^2}. \quad (4.82)$$

The spectral density of the error is expressed as:

$$F_e(\omega) = F_{in}(\omega) \cdot |K(j\omega)|^2 = \frac{2\sigma^2}{\pi T} \cdot \frac{1 - \cos \omega T}{a^2 + \omega^2}. \quad (4.83)$$

The mean square of the error is:

$$\sigma_e^2 = \int_0^\infty F_e(\omega) d\omega = \frac{2\sigma^2}{\pi T} \int_0^\infty \frac{1 - \cos \omega T}{a^2 + \omega^2} d\omega = \frac{\sigma^2}{aT} (1 - e^{-aT}). \quad (4.84)$$

The relative root mean square error is defined as follows:

$$\eta = \frac{\sigma_e}{\sigma} = \sqrt{\frac{1}{aT} (1 - e^{-aT})}. \quad (4.85)$$

We introduce the pass band  $\Delta f_{0.7}$  of the circuit, by which we shall understand the frequency band in which the modulus of its transfer ratio (3.130) is not less than  $1/\sqrt{2}$ . It can be readily seen that to the upper limit of this band corresponds the angular frequency  $\omega_{0.7} = a$ , whence we have:

$$\Delta f_{0.7} = \frac{a}{2\pi}. \quad (4.86)$$

Now, expression (4.85) can be brought into following form:

$$\eta = \sqrt{\frac{1}{2\pi \Delta f_{0.7} T} (1 - e^{-2\Delta f_{0.7} T})}. \quad (4.87)$$

If the pass band of the filter  $\Delta f_{0.7} = 0$ , then the relative error becomes  $\eta = 1$ , i. e., the signal is not reproduced at the output of the circuit. At  $\Delta f_{0.7} \rightarrow \infty$  we have  $\eta \rightarrow 0$ , i. e., the error tends to zero. The dependence of the relative error on the product  $\Delta f_{0.7} T$  is shown in Table 1.

Table 1

$\Delta f_{0.7} T$	0.0	0.1	0.2	0.3	0.4	0.5
$\eta$	1.00	0.87	0.76	0.67	0.59	0.55

At  $\Delta f_{0.7} T > 0.5$  the value of  $\eta$  can be calculated by the approximate formula

$$\eta = \frac{1}{\sqrt{2\pi\Delta f_{0.1}T}}. \quad (4.88)$$

with an error of less than 2%.

Taking  $\eta = 0.3$  (about 10% of the signal power), we obtain from (4.88):

$$\Delta f_{0.1}T \approx 1.8. \quad (4.89)$$

It is generally assumed that, for a satisfactory reproduction of the form of a telegraph signal with continually alternating signs of the pulses, the filter must let through the third harmonic of the basic frequency of the signal. The basic frequency of such a signal is equal to  $1/2T$ . Thus we obtain:

$$3 \frac{1}{2T} = \Delta f_{0.1} \quad \text{or} \quad \Delta f_{0.1}T = 1.5, \quad (4.90)$$

i. e., a result which is close to (4.89).

Let us now consider the mean square error of such a filter, which results if, in addition to the useful signal, an uncorrelated noise of spectral density  $F_n$  is applied to its input. Since the signal and the noise represent processes which are statistically independent of each other, the mean square  $\sigma_{\text{no}}^2$  of the resulting error can be expressed, in view of (3.118), (3.135) and (4.84) as follows:

$$\sigma_{\text{no}}^2 = \frac{\sigma^2}{aT} (1 - e^{-aT}) + \frac{\pi}{2} a F_n, \quad (4.91)$$

whence we obtain for the relative root mean square error

$$\eta_0 = \frac{\sigma_{\text{no}}}{\sigma} = \sqrt{\frac{1}{aT} (1 - e^{-aT}) + aTA}, \quad (4.92)$$

where

$$A = \frac{\pi}{2} \frac{F_n}{\sigma^2 T}. \quad (4.93)$$

Let us trace the connection between the resulting error and the pass band width, i. e., the quantity  $aT$ . If the pass band is narrow, the main component of the error is produced by the distortions, caused by the filter, as the mean square of the noise at the output is small. Conversely, if the pass band is wide, the distortions mentioned above are insignificant, but the mean square of the noise voltage at the output increases in proportion to the band width.

From these considerations one could expect that it is possible, under certain conditions, to realize a most advantageous pass band, which would ensure a minimum value of error. Referring to the extremal values of expression (4.92), the minimum of the relative error  $\eta_0$  can be readily obtained for the condition

$$A = \frac{1 - (1 + aT)e^{-aT}}{(aT)^2}. \quad (4.94)$$

From this expression Table 2 has been constructed for optimal values of  $aT$ .

Table 2

$(aT)_{\text{opt}}$	0	1	2	3	5
$A$	1.00	0.26	0.15	0.089	0.038

At  $(\alpha T)_{\text{opt}} > 5$  one can set, with an error of less than 4%:

$$A = \frac{1}{(\alpha T)_{\text{opt}}^2}, \quad (4.95)$$

whence. It can be con

$$(\alpha T)_{\text{opt}} = \frac{1}{\sqrt{A}}. \quad (4.96)$$

It can be concluded from these results that the higher the relative noise level  $A$  at the input of the filter, the narrower the optimal pass band. At  $A > 1$  no optimal band exists.

In view of relation (4.89), to which corresponds  $\alpha T = 11.3$ , we conclude that large values of  $\alpha T$ , for which formula (4.96) is valid, are of practical interest. Substituting the value for  $\alpha T$  obtained from (4.96) into (4.92), and taking into consideration that for  $\alpha T > 5$  the exponential term can be neglected, we obtain:

$$\eta_{\text{min}} = \sqrt{2} \sqrt[4]{A}. \quad (4.97)$$

Given the required value of  $\eta_{\text{min}}$  we can find from this expression the corresponding value of  $A$ . With known level of the noise and the desired duration of the signal, i.e., the speed of transmission, this permits to obtain, with the help of (4.93), the necessary level of the useful signal at the input, and also to find from (4.96) the optimal pass band of the filter.

The foregoing was an example of the analysis of a specific system, from the point of view of the root mean square error, suffered in it by a signal. The criterion of root mean square error was introduced into the theory of random processes by A. N. Kolmogorov /15, 16/ and was applied to a number of practical problems by N. Wiener /17/. At the present time, this criterion of quality of a system finds an ever increasing application, especially in the theory of automatic control /18, 19/. But it should be in no way considered universal. This is seen, for instance, from the following example. In pulse radar for long distance detection it is desirable to be able to detect the weakest reflected pulses against the background of fluctuation noises. Here the distortion of the pulse shape at reception is a secondary factor only, as it is first of all important to ensure that the peak value of the signal should surpass the noise as much as possible.

#### § 24. The Optimal Filter Problem

The linear problems of the theory of random processes, as considered in previous sections, are characterized by the following statement of the problem: given a specific system and its parameters, as well as a sufficiently complete statistical characterization of the input; to calculate the responses of the system to this input. These problems may be classified as belonging to the analysis of systems under random excitation.

In all cases, except the one considered in §23, only the response to random input was considered, and no conclusions about the desirable values of the parameters of these systems were drawn. But from the point of view of classification of problems this fact is not essential.

Besides the problems of analysis, problems of synthesis of systems subject to random excitation are also of great interest to the modern technology of automatic control. In these problems the structure of the system is not defined, and the totality of all possible systems of a certain class (e.g., the class of linear systems) is considered. It is required to choose from this class the

optimal system (e. g., from the point of view of minimum root mean square error). This problem was first considered in its general formulation by A. N. Kolmogorov /16/. Results important for practical applications were also obtained by N. Wiener /17/. A clear and comparatively simple presentation of the corresponding mathematical questions was given in the comprehensive article by A. M. Yaglom /3/. The same problem was considered in the monograph of V. V. Solodovnikov /19/ and in the later works of V. S. Pugachev /20, 21, 22, 23/. We will give here only one simple particular example for the synthesis of an optimal system, which ensures minimum root mean square error of reproduction for a useful signal of specific shape and uncorrelated fluctuations at the input.

As the signal we take an input with the following properties. The voltage  $u_c(t)$  can take either of two discrete values  $+U$  and  $-U$ . The mean number of changes of sign of the signal per unit of time is denoted by  $n$ . The instants of sign change are random, and the probability for  $k$  sign changes during the period  $\tau$  follows the Poisson distribution

$$W(k) = \frac{(n\tau)^k}{k!} e^{-n\tau}. \quad (4.98)$$

We remark, that the Poisson law is met with in many statistical problems, in particular in the theory of electron emission by hot cathodes. If the mean number of electrons emitted by the cathode in unit time is  $n$ , and the individual electrons are equally likely to leave the cathode at any time and do so independently of each other, then the probability for emission of  $k$  electrons by the cathode in time  $\tau$  is expressed by the same formula (4.98).

Let us find the second moment of the signal. The value of the product  $u_c(t) \cdot u_c(t+\tau)$  is equal to  $U^2$ , if during the time  $\tau$  the number of sign changes has been even, and equal to  $-U^2$  for an uneven number. We have, therefore,

$$m_2^{(c)}(\tau) = U^2 [W(0) + W(2) + W(4) + \dots] - U^2 [W(1) + W(3) + W(5) + \dots] \quad (4.99)$$

or, substituting the value of  $W(k)$  from (4.98), we obtain

$$m_2^{(c)}(\tau) = U^2 e^{-n\tau} \left( 1 - \frac{n\tau}{1!} + \frac{(n\tau)^2}{2!} - \frac{(n\tau)^3}{3!} + \dots \right) = U^2 e^{-2n\tau}. \quad (4.100)$$

Let us now formulate our problem. At the input of the linear electrical system acts a stationary random excitation consisting of the signal  $u_c$  and the fluctuation noise  $u_n$ :

$$u_1(t) = u_c(t) + u_n(t). \quad (4.101)$$

The second moment of the signal is given by the expression (4.100) and the noise is uncorrelated. The voltage  $u_2(t)$  at the output of the system reproduces the signal  $u_c(t)$  with a certain error

$$e(t) = u_2(t) - u_c(t), \quad (4.102)$$

whose mean square is equal to

$$\sigma_e^2 = \sigma_2^2 + \sigma_c^2 - 2m_2^{(c)}(0). \quad (4.103)$$

where  $m_2^{(2c)}(0)$  is the mixed moment of the input signal voltage and the output voltage, calculated for the two voltages considered as functions of the same time argument.

The mean square error  $\sigma_e^2$  is determined, on the one hand by the external influences, and on the other hand by the structure and the parameters of the system which are unknown. The problem consists in finding the structure and the parameters of the linear system which would ensure a minimum mean square error under given external conditions.

Let us set up the explicit expression for the mean square error. The mean square of the voltage at the output of the system will be determined by formula (3.30) in which we set  $t_1 = t_2 = 0$  and the lower integration limits as equal to  $-\infty$ :

$$\sigma_e^2 = \int_{-\infty}^0 \int_{-\infty}^0 m_2^{(1)}(t'_1 - t'_2) \xi(-t'_1) \xi(-t'_2) dt'_1 dt'_2. \quad (4.104)$$

By change of integration variables  $t'_1 = -x$ ,  $t'_2 = -y$ , equation (4.104) assumes the following form:

$$\sigma_e^2 = \int_0^\infty \xi(y) dy \int_0^\infty m_2^{(1)}(y - x) \xi(x) dx. \quad (4.105)$$

For computing the mixed moment  $m_2^{(2c)}(0)$  let us write the integral transformation which determines the response of the system

$$u_2(0) = \int_{-\infty}^0 u_1(t) \xi(-t) dt = \int_0^\infty u_1(x) \xi(x) dx. \quad (4.106)$$

The moment  $m_2^{(2c)}(0)$  can be considered as the mixed moment of two linear transforms of random functions, the first of which being defined by the expression (4.106) and the second consisting of the product of the random function  $u_c(t)$  and unity. Having in view the general formula (2.73) we obtain:

$$m_2^{(2c)}(0) = \int_0^\infty m_2^{(1c)}(x) \xi(x) dx, \quad (4.107)$$

where  $m_2^{(1c)}(0)$  is the mixed moment of the voltages  $u_1(t)$  and  $u_c(t)$ , which, because of the statistical independence of the signal and the noise, is equal to

$$m_2^{(1c)}(x) = m_2^{(c)}(x). \quad (4.108)$$

In view of equations (4.105), (4.107) and (4.108), we can now write the mean square of the error in the following way:

$$\begin{aligned} \sigma_e^2 = \sigma_c^2 - 2 \int_0^\infty m_2^{(c)}(x) \xi(x) dx + \\ + \int_0^\infty \xi(y) dy \int_0^\infty m_2^{(1)}(y - x) \xi(x) dx. \end{aligned} \quad (4.109)$$

Relation (4.109) is valid for any form of the moment functions of signal and noise. Only their statistical independence is necessary, in which case follows (4.108) which we used above.

The mean square of the error (4.109) depends on the form of the impulse characteristic  $\xi(x)$  of the system. First one has to find such a function  $\xi(x)$ , for which the quantity  $\sigma_e^2$  has a minimum. Problems of this kind are dealt with by the variational calculus, whose methods we shall use. Then, once the impulse characteristic of the optimal system is known, one has to determine the structure and the parameters of this system.

For determination of the impulse characteristic of the optimal system we shall proceed in the following way. Let us assume that the minimum of the mean square error is obtained for the impulse characteristic  $\xi_0(x)$ . We shall replace in the expression (4.109) the function  $\xi(x)$  by the function  $\xi_0(x) + \gamma\eta(x)$ , where  $\eta(x)$  is some function which vanishes for  $x < 0$  and is otherwise arbitrary, and  $\gamma$  is a parameter independent of  $x$ . The indicated operation of the calculus of variations is equivalent to an increase of  $x$  by  $\Delta x$  in differential calculus.

After this substitution it is easily found from (4.109) that the departure [variation] of the mean square error from its minimum equals

$$\begin{aligned} \delta\sigma_e^2 = 2\gamma \left\{ - \int_0^\infty m_2^{(c)}(x) \eta(x) dx + \right. \\ \left. + \int_0^\infty \eta(y) dy \int_0^\infty m_2^{(1)}(y-x) \xi_0(x) dx \right\} + \\ + \gamma^2 \int_0^\infty \eta(y) dy \int_0^\infty m_2^{(1)}(y-x) \eta(x) dx. \end{aligned} \quad (4.110)$$

The necessary condition for the extremum of the quantity  $\sigma_e^2$  is analogous to the corresponding condition for extrema of functions of one independent variable, and it has the form

$$\left[ \frac{d(\delta\sigma_e^2)}{d\gamma} \right]_{\gamma=0} = 0 \quad (4.111)$$

for an arbitrary function  $\eta(x)$ .

By carrying out operation (4.111), we obtain:

$$\int_0^\infty m_2^{(c)}(x) \eta(x) dx - \int_0^\infty \eta(y) dy \int_0^\infty m_2^{(1)}(y-x) \xi_0(x) dx = 0 \quad (4.112)$$

or

$$\int_0^\infty \eta(y) dy \left[ m_2^{(c)}(y) - \int_0^\infty m_2^{(1)}(y-x) \xi_0(x) dx \right] = 0. \quad (4.113)$$

Since the function  $\eta(x)$  is arbitrary, condition (4.113) can be fulfilled only for

$$m_2^{(c)}(y) = \int_0^\infty m_2^{(1)}(y-x) \xi_0(x) dx \quad (0 < y < \infty). \quad (4.114)$$

Such a condition is necessary for the mean square error to be a minimum. It can be proved that this condition is also sufficient.

Equation (4.114) is Fredholm's integral equation of the first kind. For the general case it is relatively difficult to obtain its solution. Therefore, we terminate our calculation of the general case by the obtained result, and turn our attention to the specific problem, as formulated at the beginning of this section, for which a solution can be obtained by simpler means.

The moment  $m_2^{(c)}$  is expressed by formula (4.10)). The moment  $m_2^{(1)}$  of the input voltage consists, because of the statistical independence of the signal

and the noise, of the sum of their moments, which we can write in view of expressions (3.128) and (4.100), in the following way:

$$m_2^{(1)}(y-x) = U^2 e^{-\beta|y-x|} + S\delta(y-x), \quad (4.115)$$

where  $\beta = 2n$ . Let us substitute in the integral equation (4.114) the indicated values for the moments. Then, in view of

$$\int_0^\infty S\delta(y-x)\xi_0(x)dx = S\xi_0(y), \quad (4.116)$$

we have:

$$U^2 e^{-\beta y} = S\xi_0(y) + U^2 \int_0^\infty e^{-\beta|y-x|}\xi_0(x)dx. \quad (4.117)$$

Equation (4.117) is Fredholm's integral equation of the second kind. Let us apply to both members of this equation the Laplace transformation. The transform of the definite integral, in the right-hand member of (4.117), is equal to

$$\begin{aligned} \int_0^\infty e^{-py} dy \int_0^\infty e^{-\beta|y-x|}\xi_0(x)dx &= \\ &= \int_0^\infty \xi_0(x)dx \int_0^\infty e^{-\beta|y-x|} e^{-py} dy = \\ &= \int_0^\infty \xi_0(x)dx \left\{ \int_0^x e^{-\beta(x-y)} e^{-py} dy + \int_x^\infty e^{-\beta(y-x)} e^{-py} dy \right\} = \\ &= \frac{1}{p-\beta} \int_0^\infty e^{-\beta x} \xi_0(x)dx - \frac{2\beta}{p^2-\beta^2} \int_0^\infty e^{-px} \xi_0(x)dx = \\ &= \frac{1}{p-\beta} \bar{\xi}_0(\beta) - \frac{2}{p^2-\beta^2} \bar{\xi}_0(p). \end{aligned} \quad (4.118)$$

Taking (4.118) into account, we obtain the following operational equation:

$$\frac{U^2}{p+\beta} = S\bar{\xi}_0(p) + \frac{U^2}{p-\beta} \bar{\xi}_0(\beta) - \frac{2\beta U^2}{p^2-\beta^2} \bar{\xi}_0(p), \quad (4.119)$$

whence

$$\bar{\xi}_0(p) = \frac{U^2}{S} [1 - \bar{\xi}_0(\beta)] \frac{p-\beta \frac{1+\bar{\xi}_0(\beta)}{1-\bar{\xi}_0(\beta)}}{p^2-\beta^2-2\beta \frac{U^2}{S}}. \quad (4.120)$$

From the list of formulas of operational calculus we take the relation

$$\frac{p-a}{p^2-b^2} \rightarrow \operatorname{ch} bt - \frac{a}{b} \operatorname{sh} bt = \frac{1}{2} \left(1 - \frac{a}{b}\right) e^{bt} + \frac{1}{2} \left(1 + \frac{a}{b}\right) e^{-bt}. \quad (4.121)$$

As the required impulse characteristic has to satisfy the condition  $\xi(\infty) = 0$ , the function of which (4.120) is a transform cannot contain a term which would increase exponentially with increasing argument. Therefore, in (4.121), a must equal b, which corresponds in (4.120) to

$$\beta \frac{1+\bar{\xi}_0(\beta)}{1-\bar{\xi}_0(\beta)} = \sqrt{\beta^2 + 2\beta \frac{U^2}{S}}. \quad (4.122)$$

Relation (4.122) permits the elimination of the quantity  $\bar{\xi}_0(\beta)$  from the equation (4.120) and thus to simplify it as follows:

$$\bar{\xi}_0(p) = \frac{\beta k}{\sqrt{1+k}} \frac{1}{p + \beta \sqrt{1+k}}, \quad (4.123)$$

where

$$k = \frac{2U^2}{\beta S}. \quad (4.124)$$

Let us recall that the quantity  $\beta$  entering (4.123) and (4.124) is equal to double the average number of sign changes per time unit.

By transforming (4.123) back to the original, we obtain

$$\xi_0(t) = \frac{\beta k}{\sqrt{1+k}} e^{-\beta \sqrt{1+k} t} = \left[ \frac{k}{1+k} \right] \beta \sqrt{1+k} e^{-\beta \sqrt{1+k} t}. \quad (4.125)$$

Thus, the required impulse characteristic of the optimal system has been found. Knowing it, the structure and the parameters of the system can be determined. Neglecting the factor in the square brackets of (4.125) and comparing (4.125) with (3.126), we can conclude that the RC circuit with the parameter  $\alpha$  equal to

$$\alpha = \beta \sqrt{1+k}. \quad (4.126)$$

has the impulse characteristic (4.125).

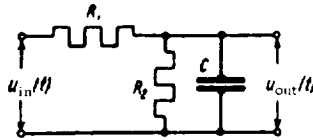


Figure 17. First circuit of an optimal filter for an irregular telegraph signal and uncorrelated fluctuation noise

The presence of the above-mentioned factor means that a voltage divider must be introduced into the circuit. In this way the circuit of Figure 17 is obtained. Its parameters are to be chosen in accordance with the conditions

$$\frac{R_2}{R_1 + R_2} = \frac{k}{1+k}, \quad (4.127)$$

$$\frac{1}{C} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) = \beta \sqrt{1+k}. \quad (4.128)$$

Assuming arbitrarily one of the parameters, the two others can easily be found from (4.127) and (4.128). Of course, the shown structure of the optimal system is not the only one possible. In particular, the same result is obtained with the circuit of Figure 18.

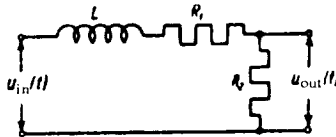


Figure 18. Second circuit of an optimal filter for an irregular telegraph signal and uncorrelated fluctuation noise

The task of the above-considered optimal system was to reproduce, in the presence of interference, the input signal with minimum mean square error. In the theory of automatic control a more general problem is also considered, where a system operating under the same conditions has to perform with minimum of error a certain transformation of a signal; the line of reasoning in this case is analogous to the foregoing.

## § 25. Elements of the Theory of Potential Noise-Stability

The concept of optimal system, as introduced in the preceding section, is in no way universal. In the following a different formulation of the problem is considered, in which the concept of optimal system acquires a completely different sense. This theory, which will be sketched in the following, was developed by V. A. Kotel'nikov [24].

Let us consider a communication channel under the action of random interference. The method of technically realizing the channel is of no importance for the following. We shall assume that for the transmission of the information through the channel two kinds of signals of equal duration  $T$ , denoted by  $A(t)$  and  $B(t)$  are used. An example of such a transmission of information is telegraphy using the Baudot code, where one of the signals is switching the current on, and the other is the interval or switching on a current of opposite sign. The mode of transmission is assumed to be known, i. e., the determinate functions  $A(t)$  and  $B(t)$  are given.

The input of the apparatus which is placed at the output of the channel—it will be called in the following receiver—is acted upon by  $X(t)$ , being at any moment the sum of one of the two indicated signals and of the interference  $W(t)$ , which is a random function of time. Thus, it is either

$$X(t) = A(t) + W_1(t), \quad (4.129)$$

or

$$X(t) = B(t) + W_2(t), \quad (4.130)$$

where  $W_1(t)$  and  $W_2(t)$  are distinct realizations of the random function  $W(t)$ .

When receiving the signal  $X(t)$ , the receiver must respond to one of the two signals  $A(t)$  and  $B(t)$  which constitute its input. The basis for the response is the comparison between the disturbed signal  $X(t)$  and the signals  $A(t)$  and  $B(t)$ , which are in some way applied to the receiver. The reaction of the receiver is determined by the closeness of  $X(t)$  to one of the signals  $A(t)$  and  $B(t)$ . The concept of closeness here used is conditioned by internal characteristics of the receiver.

As the various realizations of the disturbance may vary greatly in form, it may happen that the input  $X(t)$ , containing the signal  $A(t)$ , is closer to the signal  $B(t)$ . In such a case the response of the receiver will be false. The receiver may also respond to an  $X(t)$  containing  $B(t)$  as if it were the signal  $A(t)$ . Such errors cannot be eliminated in principle, i. e., a certain percentage of response errors is unavoidable. The probability of error depends on the internal structure of the receiver, i. e., on the concept of closeness of two functions it employs.

Following V. A. Kotel'nikov we shall call a receiver ideal, for which the probability of error is minimum. This constitutes the optimal system, whose properties are studied by the theory of potential noise-stability.

First we have to clarify the concept of closeness which is needed for the receiver to be ideal in the above-mentioned sense. We shall assume that the interference is a stationary random function of time, with normal distribution

$$w(W) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{W^2}{2\sigma^2}}. \quad (4.131)$$

After receiving the signal  $X(t)$  we can say that in the given case the interference had appeared in one of the two possible realizations, either

$$W_1(t) = X(t) - A(t), \quad (4.132)$$

or

$$W_2(t) = X(t) - B(t). \quad (4.133)$$

If the realization  $W_1(t)$  is more probable, then it is also more probable that the input  $X(t)$  consists of the signal  $A(t)$  and the interference. In the opposite case it is more likely that  $X(t)$  contains  $B(t)$ . Therefore, it becomes necessary to consider the probabilities of the different realizations of the interference.

Let us assume, for simplicity, that the interference is an uncorrelated random function of time. Then, by virtue of the statistical independence of its ordinates  $y_1, y_2, \dots, y_n$  the  $n$ -dimensional probability density of these ordinates will be

$$w(y_1, y_2, \dots, y_n) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \exp \left[ -\frac{1}{2\sigma^2} \sum_{i=1}^n y_i^2 \right], \quad (4.134)$$

where  $n$  can be arbitrarily large. This probability density is the larger, the smaller the sum of the squares of the ordinates, or, in other words, the smaller the mean square of the interference the more probable it becomes. Therefore if the inequality

$$\frac{1}{T} \int_0^T (X - A)^2 dt < \frac{1}{T} \int_0^T (X - B)^2 dt, \quad (4.135)$$

holds, it is more probable that the signal  $A(t)$  is the one transmitted. If

$$\frac{1}{T} \int_0^T (X - A)^2 dt > \frac{1}{T} \int_0^T (X - B)^2 dt, \quad (4.136)$$

then it is more probable that the input  $X(t)$  corresponds to the signal  $B(t)$ . By definition the probability of error in an ideal receiver must be minimal. Therefore, the closeness of two functions must be estimated in the ideal receiver by the mean square of their difference.

Let us consider the condition under which the ideal receiver gives a false response. Let  $A(t)$  be the signal at the input of the communication channel, i.e.,

$$X(t) = A(t) + W(t). \quad (4.137)$$

The response of the receiver to the signal (4.137) depends on which of the two inequalities (4.135) and (4.136) holds. This depends on the form of the realization of the interference in the given case. Suppose that form of the interference is such that the inequality (4.136) holds. Then the response of the receiver is false. We obtain the condition for this error by substituting into (4.136) the true value of the function  $X(t)$ , given by (4.137). Thus, the condition for the error of the ideal receiver is

$$\frac{1}{T} \int_0^T W^2(t) dt > \frac{1}{T} \int_0^T [A(t) + W(t) - B(t)]^2 dt. \quad (4.138)$$

After some simple transformations we obtain this condition in its final form:

$$\int_0^T W(t) |B(t) - A(t)| dt > \frac{1}{2} \int_0^T [A(t) - B(t)]^2 dt. \quad (4.139)$$

The probability of error of an ideal receiver is equal to the probability of fulfillment of inequality (4.139).

If the method of transmission, i. e., the functions  $A(t)$  and  $B(t)$  are given, then the right-hand part of (4.139) is some constant function. The left-hand part of (4.139) represents an integral transformation of the probability function  $W(t)$ . From the assumptions made concerning  $W(t)$  we obtain that the left-hand part is a normal random variable whose mean square, according to (3.132), is

$$\sigma^2 = S \int_0^T [A(t) - B(t)]^2 dt. \quad (4.140)$$

Therefore, the probability of error of an ideal receiver becomes

$$P_{er} = \int_M^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2\sigma^2}} dz, \quad (4.141)$$

where

$$M = \frac{1}{2} \int_0^T [A(t) - B(t)]^2 dt. \quad (4.142)$$

We shall introduce into (4.141) a new variable of integration

$$x = \frac{z}{\sigma}. \quad (4.143)$$

Then we obtain:

$$P_{er} = \frac{1}{\sqrt{2\pi}} \int_N^\infty e^{-\frac{1}{2}x^2} dx = \Psi(N), \quad (4.144)$$

where

$$N = \frac{1}{2\sqrt{S}} \sqrt{\int_0^T [A(t) - B(t)]^2 dt}. \quad (4.145)$$

The value of the function  $\Psi(N)$  is easily found from tables of the Gaussian error integral  $\Phi(x)$ . Some values of  $\Psi(N)$  are given in Table 3:

Table 3

$N$	0	0.5	1	2	3	5
$\Psi(N)$	0.5000	0.3085	0.1587	0.0228	0.00135	$3 \cdot 10^{-8}$

The calculation of the probability of error is very simple. Knowing the method of transmission (the functions  $A(t)$  and  $B(t)$ ) and the intensity of the interference (the quantity  $S$ ), integral (4.145) can be calculated, and the probability of error of the ideal receiver can then be read off Table 3.

The probability of error of an ideal receiver is a characteristic of the employed method of transmission. It shows how far, for a given transmission method, the ideal

receiver can withstand interference, or, in other words, it characterizes the potential noise-stability of this method of transmission. In a real receiver this stability is lower than the potential one. The difference in noise-stability of an ideal and a real receiver is a measure of the perfection of the latter.

Let us apply the obtained results to telegraphy, employing pulses of both signs of amplitude  $U$  and duration  $T$ . The quantity  $N$  in this case is equal to

$$N = U \sqrt{\frac{T}{S}}. \quad (4.146)$$

To increase the noise-stability larger values of  $N$  should be sought. By formula (4.146), this can be obtained in our case by increasing the amplitude  $U$  and the duration  $T$  and by decreasing the interference intensity  $S$ . It should be noted that for  $S \rightarrow \infty$  we have  $N \rightarrow 0$ , i. e.,  $P_{err} \rightarrow 0.5$ . Thus, at extremely strong interference the receiver responds correctly and falsely with equal probability. This means that one could as well dispense with calculations of the received signal  $X(t)$  and decide on which signal of the two has been transmitted by tossing a coin.

## Chapter Five

### RANDOM FORCE ON A NONLINEAR SYSTEM

#### § 26. Simplest Problem of Random Force on an Inertialess Nonlinear System

The simplest problem of a random force on a nonlinear system, which is the subject of the present section, can be formulated as follows. Let a random function  $x = x(t)$ , describing the external force on the system, be given. As was shown in the second chapter, this means that for any  $n$ , an  $n$ -dimensional probability density  $w(x_1, x_2, \dots, x_n)$  is known, where the  $x_1, x_2, \dots, x_n$  are the values of the random function at the times  $t_1, t_2, \dots, t_n$ .

Furthermore, the determinate function  $y = f(x)$  is known, which characterizes the nonlinear system under random force. The variable  $y$  is the response of the system to the external force at some instant  $t$ ; it is determined only by the magnitude of this force at the same instant and is independent of the previous course of the process. This property of the system is characterized by the term "inertialess".

The problem consists in finding under given conditions the  $n$ -dimensional probability density  $w(y_1, y_2, \dots, y_n)$  for the values  $y_1, y_2, \dots, y_n$  of the responses of the system at the times  $t_1, t_2, \dots, t_n$ .

Let us find the probability for the simultaneous fulfillment of the following inequalities:

$$\left. \begin{aligned} y_1^* - \frac{1}{2} dy_1 &\leq y_1 \leq y_1^* + \frac{1}{2} dy_1, \\ y_2^* - \frac{1}{2} dy_2 &\leq y_2 \leq y_2^* + \frac{1}{2} dy_2, \\ &\dots\dots\dots \\ y_n^* - \frac{1}{2} dy_n &\leq y_n \leq y_n^* + \frac{1}{2} dy_n. \end{aligned} \right\} \quad (5.1)$$

This probability is equal to  $w(y_1^*, y_2^*, \dots, y_n^*) dy_1, dy_2, \dots, dy_n$ . Therefore, if the joint probability of fulfillment of the inequalities (5.1) is known, we obtain immediately the required probability density  $w(y_1^*, y_2^*, \dots, y_n^*)$ .

Let us characterize the set of  $n$  numbers  $x_1, x_2, \dots, x_n$  by a point of  $n$ -dimensional space with the coordinates  $x_1, x_2, \dots, x_n$ . We shall call it the representing point. Then it can be said that the responses  $y_1, y_2, \dots, y_n$  are functions of the position of the representing point in the  $n$ -dimensional space.

All inequalities (5.1) are satisfied if the representing point is contained in a

certain region D of the n-dimensional space. The probability for the fulfillment of these inequalities is equal to the probability for the representing point to be in the region D. Therefore the required probability can be written as

$$\begin{aligned} w(y_1^*, y_2^*, \dots, y_n^*) dy_1 dy_2 \dots dy_n = \\ = \int \dots \int_D w(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n, \end{aligned} \quad (5.2)$$

with the integral taken over the region D.

The right-hand side of (5.2) can be calculated without knowing the boundary of the region D by the following device. We introduce under the integral the factor  $\Delta(x_1, x_2, \dots, x_n)$  which is equal to unity if the representing point is in the region D, and equal to zero if it lies outside this region. Then integration over the region D can be replaced by integration over the entire n-dimensional space and equation (5.2) can be written as follows:

$$\begin{aligned} w(y_1^*, y_2^*, \dots, y_n^*) dy_1 dy_2 \dots dy_n = \\ = \int \dots \int_{-\infty}^{+\infty} \Delta(x_1, x_2, \dots, x_n) w(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n. \end{aligned} \quad (5.3)$$

The factor  $\Delta$  can be explicitly expressed by means of the Dirichlet integral

$$\delta = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin \pi z}{z} e^{j\pi z} dz, \quad (5.4)$$

which is equal to unity if  $z < \gamma < \alpha$ , and equal to zero if  $\gamma$  is outside this interval. We shall set in (5.4)

$$\alpha = \frac{1}{2} dy_1 \quad (5.5)$$

and

$$\gamma = y_1 - y_1^* = f(x_1) - y_1^*. \quad (5.6)$$

Then we obtain from (5.4)

$$\delta_1 = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin\left(\frac{1}{2} dy_1 \cdot z_1\right)}{z_1} \exp[jz_1 \{f(x_1) - y_1^*\}] dz_1. \quad (5.7)$$

Since

$$\sin\left(\frac{1}{2} dy_1 \cdot z_1\right) = \frac{1}{2} dy_1 \cdot z_1, \quad (5.8)$$

equation (5.7) takes the form

$$\delta_1 = \frac{dy_1}{2\pi} \int_{-\infty}^{+\infty} \exp[jz_1 f(x_1)] \exp[-jz_1 y_1^*] dz_1. \quad (5.9)$$

In view of (5.5) and (5.6) we conclude that the value of  $\delta_1$  is equal to unity if  $y_1$  is within the limits defined by the first of the inequalities (5.1) and vanishes

if  $y_1$  is outside these limits. Analogously, it is not difficult to obtain expressions similar to (5.9) for the responses  $y_2, y_3, \dots, y_n$ . The value of the factor  $\Delta$  becomes

$$\Delta(x_1, x_2, \dots, x_n) = \delta_1 \delta_2 \dots \delta_n =$$

$$= \frac{dy_1 dy_2 \dots dy_n}{(2\pi)^n} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \exp \{ j [z_1 f(x_1) + z_2 f(x_2) + \dots$$

$$\dots + z_n f(x_n)] \} \cdot \exp \{ -j (y_1^* z_1 + y_2^* z_2 + \dots + y_n^* z_n) \} dz_1 dz_2 \dots dz_n. \quad (5.10)$$

Substituting this result in (5.3) and omitting the no longer necessary asterisks from the responses  $y_1^*, y_2^*, \dots, y_n^*$ , we obtain:

$$w(y_1, y_2, \dots, y_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} A(z_1, z_2, \dots, z_n) \times$$

$$\times \exp \{ -j (y_1 z_1 + y_2 z_2 + \dots + y_n z_n) \} dz_1 dz_2 \dots dz_n, \quad (5.11)$$

where

$$A(z_1, z_2, \dots, z_n) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \exp \{ j [z_1 f(x_1) + z_2 f(x_2) + \dots$$

$$\dots + z_n f(x_n)] \} w(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n. \quad (5.12)$$

From equation (5.11) follows that the function  $A(z_1, z_2, \dots, z_n)$  is the characteristic function of the  $n$ -dimensional probability density  $w(y_1, y_2, \dots, y_n)$ :

$$A(z_1, z_2, \dots, z_n) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} w(y_1, y_2, \dots, y_n) \times$$

$$\times \exp \{ j (z_1 y_1 + z_2 y_2 + \dots + z_n y_n) \} dy_1 dy_2 \dots dy_n. \quad (5.13)$$

The obtained relations (5.11) and (5.12) determine the  $n$ -dimensional probability density  $w(y_1, y_2, \dots, y_n)$  for the general case, and they represent the required solution of the problem. From these relations the particular but important solutions for  $n = 1$  and  $n = 2$  result automatically. For  $n = 1$  we have:

$$w(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A(z) \exp \{ -j y z \} dz, \quad (5.14)$$

where

$$A(z) = \int_{-\infty}^{+\infty} \exp \{ j z f(x) \} w(x) dx. \quad (5.15)$$

For  $n = 2$  we have

$$w(y_1, y_2) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A(z_1, z_2) \exp \{ -j (y_1 z_1 + y_2 z_2) \} dz_1 dz_2, \quad (5.16)$$

where

$$A(z_1, z_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp \{ j [z_1 f(x_1) + z_2 f(x_2)] \} w(x_1, x_2) dx_1 dx_2. \quad (5.17)$$

The direct application of the general expressions (5.11) and (5.12) or of their particular cases (5.14) and (5.15) for  $n = 1$  and (5.16) and (5.17) for  $n = 2$ , leads sometimes to cumbersome calculations. If to the function  $f(x)$  corresponds a single-valued inverse function  $x = \varphi(y)$  of a sufficiently simple structure, then the following device is more expedient. We replace in (5.12) the integration variables  $x_1, x_2, \dots, x_n$  by the new integration variables  $y_1, y_2, \dots, y_n$ . Then we have:

$$A(z_1, z_2, \dots, z_n) \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \exp \{j(z_1 y_1 + z_2 y_2 + \dots + z_n y_n)\} \times \\ \times w[\varphi(y_1), \varphi(y_2), \dots, \varphi(y_n)] \varphi'(y_1) \varphi'(y_2) \dots \\ \dots \varphi'(y_n) dy_1 dy_2 \dots dy_n. \quad (5.18)$$

Combining the equations (5.13) and (5.18) we obtain the following simple result:

$$w(y_1, y_2, \dots, y_n) = \\ = w_x[\varphi(y_1), \varphi(y_2), \dots, \varphi(y_n)] \varphi'(y_1) \varphi'(y_2) \dots \varphi'(y_n), \quad (5.19)$$

where  $w_x[\varphi(y_1), \varphi(y_2), \dots, \varphi(y_n)]$  is the  $n$ -dimensional probability density for the random variables  $x_1, x_2, \dots, x_n$  under the substitution  $x_1 = \varphi(y_1), x_2 = \varphi(y_2), \dots, x_n = \varphi(y_n)$ . For the particular cases (5.15) and (5.17) we obtain, analogously, for  $n = 1$

$$w(y) = w_x[\varphi(y)] \varphi'(y), \quad (5.20)$$

and for  $n = 2$

$$w(y_1, y_2) = w_x[\varphi(y_1), \varphi(y_2)] \varphi'(y_1) \varphi'(y_2). \quad (5.21)$$

It should be noted that when the domain of existence of  $y$  does not extend from  $-\infty$  to  $+\infty$  but is smaller, the integration of (5.13) and (5.18) has to be carried out over this domain, but the validity of the equations (5.19), (5.20) and (5.21) is not affected.

In many cases the exhaustive characterization of the statistical properties of the response  $y$ , as given by the  $n$ -dimensional probability density  $w(y_1, y_2, \dots, y_n)$  is superfluous, and it is sufficient to calculate the first few moments of the response. To find these moments we proceed as follows. In the expression (5.15) for the one-dimensional characteristic function, we expand the exponential factor of the integrand in a power series:

$$\exp \{jzf(x)\} = 1 + \frac{jzf(x)}{1!} + \dots + \frac{\{jzf(x)\}^n}{n!} + \dots \quad (5.22)$$

Then the function  $A(z)$  can be written as the following power series:

$$A(z) = \sum_{v=0}^{\infty} \frac{m_v}{v!} (jz)^v, \quad (5.23)$$

where

$$m_v = \int_{-\infty}^{+\infty} \{f(x)\}^v w(x) dx. \quad (5.24)$$



For each response of the system an  $n$ -dimensional probability density must be determined. In view of the statistical dependence between the responses, an  $m \cdot n$ -dimensional joint probability density must be found for them.

The solution of the above problem does not present any fundamental difficulties. But, in order to avoid cumbersome calculations, we shall solve it below for the particular case of  $k = m = n = 2$ . The generalization of the results for any  $k$ ,  $m$  and  $n$  is obvious.

With the above limitations, the system of equations (5.29) reduces to

$$\begin{cases} y_1 = f_1(x_1, x_2) \\ y_2 = f_2(x_1, x_2) \end{cases} \quad (5.30)$$

For the external forces the four-dimensional probability density  $w(x_{11}, x_{12}, x_{21}, x_{22})$  is given, where the first index gives the ordinal number of the input, and the second shows to which of the time instants,  $t_1$  or  $t_2$ , the input corresponds. It is required to find the four-dimensional probability density  $w(y_{11}, y_{12}, y_{21}, y_{22})$ , where the indexes have the same meaning as above.

Following the method of the preceding section, let us find the probability for the simultaneous fulfillment of the following four inequalities:

$$\left. \begin{aligned} y_{11}^* - \frac{1}{2} dy_{11} &\leq y_{11} \leq y_{11}^* + \frac{1}{2} dy_{11} \\ y_{12}^* - \frac{1}{2} dy_{12} &\leq y_{12} \leq y_{12}^* + \frac{1}{2} dy_{12} \\ y_{21}^* - \frac{1}{2} dy_{21} &\leq y_{21} \leq y_{21}^* + \frac{1}{2} dy_{21} \\ y_{22}^* - \frac{1}{2} dy_{22} &\leq y_{22} \leq y_{22}^* + \frac{1}{2} dy_{22} \end{aligned} \right\} \quad (5.31)$$

which equals  $w(y_{11}^*, y_{12}^*, y_{21}^*, y_{22}^*) dy_{11} dy_{12} dy_{21} dy_{22}$ .

The values  $x_{11}$ ,  $x_{12}$ ,  $x_{21}$ , and  $x_{22}$  of the external forces will be considered as coordinates of the representing point in four-dimensional space. By analogy with (5.2) we have for the four-dimensional probability density of responses:

$$\begin{aligned} w(y_{11}^*, y_{12}^*, y_{21}^*, y_{22}^*) dy_{11} dy_{12} dy_{21} dy_{22} = \\ = \int \int \int \int_D w(x_{11}, x_{12}, x_{21}, x_{22}) dx_{11} dx_{12} dx_{21} dx_{22} \end{aligned} \quad (5.32)$$

In order to extend the integration of (5.32) over the entire space we introduce into the integrand the factor

$$\Delta = \delta_1 \cdot \delta_2 \cdot \delta_3 \cdot \delta_4. \quad (5.33)$$

In order to obtain the co-factor  $\delta_1$  we set in the Dirichlet integral (5.4):

$$\alpha = \frac{1}{2} dy_{11} \quad (5.34)$$

and

$$\gamma = y_{11} - y_{11}^* = f(x_{11}, x_{21}) - y_{11}^* \quad (5.35)$$

Then we obtain

$$\delta_1 = \frac{dy_{11}}{2\pi} \int_{-\infty}^{+\infty} \exp[jz_1 f_1(x_{11}, x_{21})] \exp[-jz_1 y_{11}^*] dz_1. \quad (5.36)$$

The co-factors  $\delta_2$ ,  $\delta_3$  and  $\delta_4$  have analogous forms. Therefore, the expression (5.33) becomes

$$\begin{aligned} \Delta(x_{11}, x_{12}, x_{21}, x_{22}) = \\ = \frac{dy_{11} dy_{12} dy_{21} dy_{22}}{16\pi^4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp[j\{z_1 f_1(x_{11}, x_{21}) + \\ + z_2 f_1(x_{12}, x_{22}) + z_3 f_2(x_{11}, x_{21}) + z_4 f_2(x_{12}, x_{22})\}] \times \\ \times \exp[-j(y_{11}^* z_1 + y_{12}^* z_2 + y_{21}^* z_3 + y_{22}^* z_4)] dz_1 dz_2 dz_3 dz_4. \end{aligned} \quad (5.37)$$

Introducing the factor  $\Delta$  into the integrand function (5.32) and omitting the asterisks from the responses  $y$ , we obtain:

$$\begin{aligned} w(y_{11}, y_{12}, y_{21}, y_{22}) = \\ = \frac{1}{16\pi^4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A(z_1, z_2, z_3, z_4) \exp[-j(y_{11} z_1 + \\ + y_{12} z_2 + y_{21} z_3 + y_{22} z_4)] dz_1 dz_2 dz_3 dz_4, \end{aligned} \quad (5.38)$$

where

$$\begin{aligned} A(z_1, z_2, z_3, z_4) = \\ = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp[j\{z_1 f_1(x_{11}, x_{21}) + z_2 f_1(x_{12}, x_{22}) + \\ + z_3 f_2(x_{11}, x_{21}) + z_4 f_2(x_{12}, x_{22})\}] \times \\ \times w(x_{11}, x_{12}, x_{21}, x_{22}) dx_{11} dx_{12} dx_{21} dx_{22}. \end{aligned} \quad (5.39)$$

Here the function  $A(z_1, z_2, z_3, z_4)$  is the characteristic function of the four-dimensional probability density  $w(y_{11}, y_{12}, y_{21}, y_{22})$ :

$$\begin{aligned} A(z_1, z_2, z_3, z_4) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} w(y_{11}, y_{12}, y_{21}, y_{22}) \cdot \exp[j(z_1 y_{11} + \\ + z_2 y_{12} + z_3 y_{21} + z_4 y_{22})] dy_{11} dy_{12} dy_{21} dy_{22}. \end{aligned} \quad (5.40)$$

If to the system of functions (5.30) corresponds a sufficiently simple system of inverse functions

$$\left. \begin{aligned} x_1 &= \varphi_1(y_1, y_2), \\ x_2 &= \varphi_2(y_1, y_2), \end{aligned} \right\} \quad (5.41)$$

then the probability density  $w(y_{11}, y_{12}, y_{21}, y_{22})$  can be found more simply in the following way. Let us replace in equation (5.39) the integration variables  $x_{11}, x_{12}, x_{21}$ , and  $x_{22}$  by new variables  $y_{11}, y_{12}, y_{21}$  and  $y_{22}$ . Then, in accordance with the rule for changing integration variables in multiple integrals, we obtain:

$$\begin{aligned} A(z_1, z_2, z_3, z_4) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp[j(z_1 y_{11} + z_2 y_{12} + z_3 y_{21} + z_4 y_{22})] \times \\ \times w[\varphi_1(y_{11}, y_{21}), \varphi_1(y_{12}, y_{22}), \varphi_2(y_{11}, y_{21}), \varphi_2(y_{21}, y_{22})] \times \\ \times |D_1 D_2| dy_{11} dy_{12} dy_{21} dy_{22}, \end{aligned} \quad (5.42)$$

where  $D_1$  and  $D_2$  are the functional determinants

$$D_1 = \begin{vmatrix} \frac{\partial \varphi_1(y_{11}, y_{21})}{\partial y_{11}} & \frac{\partial \varphi_1(y_{11}, y_{21})}{\partial y_{21}} \\ \frac{\partial \varphi_2(y_{11}, y_{21})}{\partial y_{11}} & \frac{\partial \varphi_2(y_{11}, y_{21})}{\partial y_{21}} \end{vmatrix}, \quad (5.43)$$

$$D_2 = \begin{vmatrix} \frac{\partial \varphi_1(y_{12}, y_{22})}{\partial y_{12}} & \frac{\partial \varphi_1(y_{12}, y_{22})}{\partial y_{22}} \\ \frac{\partial \varphi_2(y_{12}, y_{22})}{\partial y_{12}} & \frac{\partial \varphi_2(y_{12}, y_{22})}{\partial y_{22}} \end{vmatrix}. \quad (5.44)$$

Combining the expressions (5.40) and (5.42) we obtain the final result:

$$w(y_{11}, y_{12}, y_{21}, y_{22}) = w_x[\varphi_1(y_{11}, y_{21}), \varphi_1(y_{12}, y_{22}), \varphi_2(y_{11}, y_{21}), \varphi_2(y_{12}, y_{22})] |D_1 D_2|, \quad (5.45)$$

where  $w_x$  is the four-dimensional probability density of the inputs, with the variables changed as indicated in (5.45).

The transition from (5.39) to (5.42) and, consequently, the expression (5.45) are correct if the following conditions are complied with: 1) the functions  $\varphi_1$  and  $\varphi_2$  and their partial derivatives are continuous in the domain of integration; 2) the product of the determinants  $D_1 D_2$  does not change sign in this domain; 3) there is a one-to-one correspondence between responses and external forces. The last condition limits the range of application of formula (5.45) and of its possible generalizations to cases in which the number of the inputs of the system is equal to the number of its outputs ( $k = m$ ). Formulas obtained by generalization of (5.38) and (5.39) are free from this limitation.

To obtain the relations for the moments of the response we write the exponential factor of the integrand of (5.39) as a power series:

$$\begin{aligned} & \exp \{ [z_1 f_1(x_{11}, x_{21}) + z_2 f_1(x_{12}, x_{22}) + \\ & \quad + z_3 f_2(x_{11}, x_{21}) + z_4 f_2(x_{12}, x_{22})] \} = \\ & = \sum_{v_1, v_2, v_3, v_4} \frac{[f_1(x_{11}, x_{21})]^{v_1} [f_1(x_{12}, x_{22})]^{v_2} [f_2(x_{11}, x_{21})]^{v_3} [f_2(x_{12}, x_{22})]^{v_4}}{v_1! v_2! v_3! v_4!} \times \\ & \quad \times (jz_1)^{v_1} (jz_2)^{v_2} (jz_3)^{v_3} (jz_4)^{v_4}. \end{aligned} \quad (5.46)$$

Then (5.39) becomes:

$$\begin{aligned} A(z_1, z_2, z_3, z_4) &= \\ &= \sum_{v_1, v_2, v_3, v_4} \frac{m_{v_1+v_2+v_3+v_4}}{v_1! v_2! v_3! v_4!} (jz_1)^{v_1} (jz_2)^{v_2} (jz_3)^{v_3} (jz_4)^{v_4}, \end{aligned} \quad (5.47)$$

where the four-dimensional moments of the responses  $y_1$  and  $y_2$  are defined by

$$\begin{aligned} m_{v_1+v_2+v_3+v_4} &= \\ &= \int \int \int \int_{-\infty}^{+\infty} |f_1(x_{11}, x_{21})|^{v_1} |f_1(x_{12}, x_{22})|^{v_2} |f_2(x_{11}, x_{21})|^{v_3} \times \\ & \quad \times |f_2(x_{12}, x_{22})|^{v_4} w(x_{11}, x_{12}, x_{21}, x_{22}) dx_{11} dx_{12} dx_{21} dx_{22}. \end{aligned} \quad (5.48)$$

For the mixed moment of the fourth order we have from (5.48), for  $v_1 = v_2 = v_3 = v_4 = 1$ :

$$\begin{aligned} m_4 &= \int \int \int \int_{-\infty}^{+\infty} f_1(x_{11}, x_{21}) f_1(x_{12}, x_{22}) f_2(x_{11}, x_{21}) f_2(x_{12}, x_{22}) \times \\ & \quad \times w(x_{11}, x_{12}, x_{21}, x_{22}) dx_{11} dx_{12} dx_{21} dx_{22}. \end{aligned} \quad (5.49)$$

Our problem has thus been solved completely in principle. It should be borne in mind, however, that the application of the obtained general relationships to specific cases meets sometimes with considerable computational difficulties.

## §28. Random Processes in Inertial Nonlinear Systems

The results of the two preceding sections lead to the conclusion that the solution of the problem of random processes in inertialess, nonlinear systems does not present, in principle, any difficulties. But the calculation of the moments of the responses is much more complicated in such a system than in an inertial linear one, as much more subtle features of the input have to be taken into account.

The difficulties increase sharply if the nonlinear system possesses inertia. The reason for this can be readily grasped by taking into account that the behavior of such a system is described by a stochastic differential equation, or by a system of such equations, and considering further that at present there are no general methods of solution of nonlinear differential equations, whether for random functions or for determinate functions. Owing to this, the theory of random processes in inertial nonlinear systems is at present still in its initial stages.

The most general problem concerning a random process in an inertial nonlinear system with one input and one output (for definiteness we shall consider below only such systems) is the following. For any  $n$ , the  $n$ -dimensional probability densities of applied input are given: it is required to find the corresponding multi-dimensional probability densities for the responses of the system.

Such a problem is already difficult for an inertial linear system (with the exception of trivial cases, when the input is normally distributed, or when there is no statistical dependence between values arbitrarily near in time). Far from being complete, the results hitherto obtained are of only preliminary character (see for instance, /25/). For inertial nonlinear systems even such results are lacking.

The problem of random processes in nonlinear inertial systems is substantially simplified if the external force is uncorrelated. In such a case the Fokker-Planck equation can be used for its solution.

The Fokker-Planck equation was obtained while developing the theory of the Brownian movement. A rigorous derivation of this equation is given in the work of A. N. Kolmogorov /26/, a simplified derivation is contained in the book of M. A. Leontovich /27/, in which it is called the Einstein-Fokker equation. The method of applying the Fokker-Planck equation to the analysis of random processes in nonlinear systems is shown in the work of L. Pontryagin, A. Andronov and A. Vitt /28/.

By using the Fokker-Planck equation a number of investigators obtained important results concerning random excitation of tube oscillators. The first results in this direction were obtained by I. L. Bernstein /29, 30/. Along with the works of Bernstein, the works of G. S. Gorelik /31/ and S. M. Rytov /32/ on the same subject should be mentioned, as well as the article by P. I. Kuznetsov, R. L. Stratonovich and V. I. Tikhonov /33/. The same problem was solved in a different way by I. S. Gonorovski /34/.

Without belittling the great importance of the results obtained by the above authors, it should nevertheless be borne in mind that the case of a nonlinear system under uncorrelated action is far from exhausting all the important physical and technical problems relating to random force on nonlinear systems. Thus, a

further search for methods of analysis of random processes in nonlinear inertial systems is necessary.

The simplicity of the laws for the transformation of moments suggests that the search for laws of response distribution be abandoned and that we should be content with the calculation of its moments which, in many practical cases, give a sufficiently complete idea of the random process. But this simplified formulation of the problem, which leads to simple solutions in linear problems, turns out to be rather cumbersome for nonlinear systems. This is due to the fact that, in contrast to linear systems, where any moment of the response is determined by a moment of input of the same order, in nonlinear systems any moment of the response is determined by an infinite number of moments of the input, as was noted at the end of §26 with respect to inertialess systems. Therefore the expression for any moment must have the form of an infinite function series. Similar results were obtained by V.S. Pugachev /8/, as well as by P.I. Kuznetsov, R.L. Stratonovich and V.I. Tikhonov /35/.

While the methods of solution of the problem of calculating the moments of the response of the system as given by the above-mentioned investigators give, in principle, accurate solutions, they are very cumbersome, which greatly reduces the possibility of their practical application.

It follows from the above, that, when solving such problems one often has to abandon the use of some general methods of analysis, and one has to solve each problem approximately, by applying special methods, adapted to the peculiar features of the problem. As examples of such an approach to this question we mention the works by L.S. Gutkin /36/ and V.I. Tikhonov /37, 38/.

In conclusion, a particular class of inertial nonlinear systems should be mentioned, the analysis of the random processes occurring in which made by comparatively simple means. These are systems consisting of two mutually independent units in cascade connection, the first of which being inertialess and nonlinear, and the second inertial and linear. We call a connection "cascade" if the output of the first unit is connected to the input of the second.

In such a system the random function, equivalent to the input, is subjected first to an inertialess nonlinear transformation, and then to a linear inertial one. Knowing the distribution law of the input to the first, and using the relationships of the present chapter, the moments of the response on the output of the first, i.e., input of the second unit can be found, and then, by the methods of the third chapter, the moments of the response at the output of the second unit can be calculated.

## Chapter Six

### SOME NONLINEAR PROBLEMS IN THE THEORY OF RANDOM PROCESSES

#### §29. Action of Fluctuation Noise on a Detector with Exponential Characteristic.

In many cases the actual volt-ampere characteristic of a tube diode is well approximated by the exponential function

$$i = J_0 e^{au}, \quad (6.1)$$

where  $i$  is the anode current of the diode,  $u$  the voltage on its anode,  $J_0$  the anode current at  $u = 0$  and  $a$  is some constant parameter.

Suppose that a diode with the volt-ampere characteristic (6.1) is under the action of a fluctuation noise  $u$  (Figure 19). It is supposed that the noise voltage is stationary and has normal distribution with the mean square  $\sigma_u^2$ , i.e.

$$w(u) = \frac{1}{\sqrt{2\pi}\sigma_u} e^{-\frac{u^2}{2\sigma_u^2}}. \quad (6.2)$$

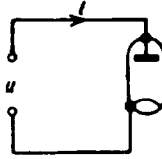


Figure 19. Diode under fluctuating voltage

We assume the normalized autocorrelation function of this voltage to be of the form

$$k_u(\tau) = e^{-\alpha\tau} \cos \omega_0 \tau. \quad (6.3)$$

Expression (6.3) corresponds to formula (4.31) for the second moment of the fluctuation voltage in an oscillation circuit, connected to a source of correlated random electromotive force. It is required to examine the statistical properties of the current in the circuit of the detector.

As the process is stationary, the d.c. component of the current in the circuit of the detector is the first moment of the current and is independent of time. Setting in (5.20)  $\nu = 1$  and taking into consideration (6.1) and (6.2) we have

$$J_{\pm} = m_{\pm} = \int_{-\infty}^{+\infty} J_0 e^{au} \frac{1}{\sqrt{2\pi\sigma_u^2}} e^{-\frac{u^2}{2\sigma_u^2}} du =$$

$$= \frac{J_0}{\sqrt{2\pi\sigma_u^2}} \int_{-\infty}^{+\infty} \exp\left[-\frac{u^2}{2\sigma_u^2} + au\right] du. \quad (6.4)$$

From the table of definite integrals [6] we take the following formula:

$$\int_{-\infty}^{+\infty} e^{-px^2+qx} dx = e^{\frac{q^2}{4p}} \sqrt{\frac{\pi}{p}}. \quad (6.5)$$

Then, after some simple transformations, we obtain from (6.4)

$$J_{\pm} = J_0 e^{\frac{1}{2} a^2 \sigma_u^2}. \quad (6.6)$$

The increase in the d. c. component of the current under the action of the noise voltage is equal to

$$\Delta J = J_{\pm} - J_0 = J_0 \left[ e^{\frac{1}{2} a^2 \sigma_u^2} - 1 \right]. \quad (6.7)$$

Let us find the second moment of the current. For that purpose we shall write down the expression for the two-dimensional probability density of the noise voltage:

$$w(u_1, u_2) = \frac{1}{2\pi\sigma_u^2 \sqrt{1-k_u^2}} \exp\left[ -\frac{u_1^2 - 2k_u u_1 u_2 + u_2^2}{2\sigma_u^2 (1-k_u^2)} \right], \quad (6.8)$$

where  $k_u = k_u(\tau)$  is the normalized autocorrelation function of the voltage acting on the detector. Now, in view of the formula (5.24), we can write

$$m_2^{(i)}(\tau) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} J_0 e^{au_1} J_0 e^{au_2} \cdot \frac{1}{2\pi\sigma_u^2 \sqrt{1-k_u^2}} \times$$

$$\times \exp\left[ -\frac{1}{2\sigma_u^2 (1-k_u^2)} (u_1^2 - 2k_u u_1 u_2 + u_2^2) \right] du_1 du_2 =$$

$$= \frac{J_0^2}{2\pi\sigma_u^2 \sqrt{1-k_u^2}} \int_{-\infty}^{+\infty} \exp\left[ -\frac{u_2^2}{2\sigma_u^2 (1-k_u^2)} + au_2 \right] du_2 \times$$

$$\times \int_{-\infty}^{+\infty} \exp\left[ -\frac{u_1^2}{2\sigma_u^2 (1-k_u^2)} + \left( \frac{k_u a}{\sigma_u^2 (1-k_u^2)} + a \right) u_1 \right] du_1. \quad (6.9)$$

The calculation of the inner and the outer integrals in this expression is easily done by applying (6.5), and gives the following simple result:

$$m_2^{(i)}(\tau) = J_0^2 \exp[a^2 \sigma_u^2 (1+k_u)] = J_{\pm}^2 e^{a^2 \sigma_u^2 k_u}. \quad (6.10)$$

At  $\tau = 0$  we have  $k_u = 1$ , and we obtain for the mean square of the current the expression

$$\sigma_i^2 = J_{\pm}^2 e^{a^2 \sigma_u^2}. \quad (6.11)$$

The mean square of the noise component of the current in the circuit of the detector is determined as follows:

$$\sigma_{in}^2 = \sigma_i^2 - J_{\pm}^2 = J_{\pm}^2 (e^{a^2 \sigma_u^2} - 1). \quad (6.12)$$

Let us examine the expression (6.10) for the second moment of the current. Replacing the exponential factor of this expression by a power series and taking into consideration (6.3) we obtain

$$m_2^{(i)}(\tau) = J_z^2 \sum_{n=0}^{\infty} \frac{(a^2 \sigma_u^2 e^{-\sigma \tau} \cos \omega_0 \tau)^n}{n!}. \quad (6.13)$$

By expressing the factor  $\cos^n \omega_0 \tau$  in each term of the series (6.13) as a sum of a constant number (for even  $n$ ) and cosines of multiples of the angle, we find that the moment  $m_2^{(i)}(\tau)$  contains an aperiodic component, which decreases with increasing  $\tau$ , and damped oscillation components of frequencies  $\omega_0, 2\omega_0, 3\omega_0, \dots$ . From the point of view of the detection process, the aperiodic component, related to the terms of the series corresponding to even values of  $n$ , is of fundamental interest.

Let us isolate this component. As well known, the constant term for  $\cos^{2m} x$  is equal to  $(2m)!/2^{2m}(m!)^2$ . Therefore, the above-mentioned component is obtained from (6.13) in the following form:

$$m_{2 \text{ a per}}^{(i)}(\tau) = J_z^2 \sum_{m=0}^{\infty} \frac{(a^2 \sigma_u^2 e^{-\sigma \tau})^{2m}}{2^{2m} (m!)^2}. \quad (6.14)$$

Taking into consideration the well-known relation from the theory of cylindrical functions

$$I_0(z) = \sum_{m=0}^{\infty} \frac{1}{(m!)^2} \left(\frac{z}{2}\right)^{2m}, \quad (6.15)$$

where  $I_0(z)$  is a modified Bessel function of zero order, we can write (6.14) in a more compact form:

$$m_{2 \text{ a per}}^{(i)}(\tau) = J_z^2 \cdot I_0(a^2 \sigma_u^2 e^{-\sigma \tau}). \quad (6.16)$$

We shall find the spectral density of the current, which corresponds to the aperiodic component of the moment in the following way. Applying the formula (3.62) to individual terms of the series (6.14) we obtain

$$\begin{aligned} \frac{2}{\pi} \int_0^{\infty} J_z^2 \frac{(a^2 \sigma_u^2 e^{-\sigma \tau})^{2m}}{2^{2m} (m!)^2} \cos \omega \tau d\tau = \\ = \frac{2}{\pi} J_z^2 \frac{(a \sigma_u)^{4m}}{2^{2m} (m!)^2} \frac{2ma}{4m^2 a^2 + \omega^2}. \end{aligned} \quad (6.17)$$

Therefore, the complete spectral density, as determined by the aperiodic component of the moment, becomes

$$F_{2 \text{ a per}}(\omega) = \frac{2}{\pi} J_z^2 \sum_{m=1}^{\infty} \frac{(a \sigma_u)^{4m}}{2^{2m} (m!)^2} \frac{2ma}{4m^2 a^2 + \omega^2}. \quad (6.18)$$

It follows from this expression that the considered component of the spectrum

of the current in the detector circuit has a maximum at  $\omega = 0$ , and decreases monotonically with increasing frequency. It can be called low-frequency component. It appears as a result of rectified noise voltage.

Analogously, we can single out from the series (6.13) damped oscillation components of the spectral density with maxima in the ranges of  $\omega_0, 2\omega_0, 3\omega_0, \dots$  respectively. We note that the spectrum of the input voltage has its only maximum in the frequency range  $\omega_0$ .

The appearance of maxima of the spectral density of the response for harmonics of the frequency of the input spectrum maximum is characteristic of random processes in nonlinear systems. The aperiodic component of the response spectrum can be missing in certain cases; this occurs if the characteristic of the nonlinear element is an odd function.

### §30. Statistical Properties of the Noise Voltage Envelope at the Output of a Selective System

It has been shown in § 21 that the noise voltage at the output of a selective system, tuned to the frequency  $\omega_0$ , is an oscillation of frequency  $\omega_0$ , modulated at random in frequency and phase. In some cases, and particularly in analyzing the action of such a voltage on the amplitude detector of a radio receiver, it is important to know the statistical properties of the random amplitude of this voltage. This question was considered in the publications of V.I. Bunimovich [4] and S. Rice [39]. We give in the following an account of the results stated in these works.

The instantaneous noise voltage on the output of a linear system is expressed by formula (3.26), in which, by limiting ourselves to the consideration of a stationary random process, we assume the lower integration limit to be equal to  $-\infty$ , and the upper limit to be  $t_1 = 0$ . This last assumption can always be made by choosing a suitable time origin. Let us denote the integration variable by  $t'$ . Then

$$u_{\text{out}}(0) = \int_{-\infty}^0 u_{\text{in}}(t') \xi(-t') dt'. \quad (6.19)$$

By changing the integration variable:  $t = -t'$ , we obtain

$$u_{\text{out}}(0) = \int_0^{\infty} u_{\text{in}}(t) \xi(t) dt. \quad (6.20)$$

Let us analyze a single oscillation circuit, connected to a source of noise electromotive force. More general results can be obtained analogously. We shall consider them later on.

The impulse characteristic of a series oscillation circuit can be expressed as follows:

$$\xi(t) = \frac{\omega_0^2}{\omega_1} e^{-\alpha t} \sin \omega_1 t \approx \omega_0 e^{-\alpha t} \sin \omega_0 t. \quad (6.21)$$

with notations as in § 21. This result can be obtained in various ways. In particular, if the transfer ratio operator of the circuit is obtained from (4.22) by the substitution  $j\omega = p$ , and considering that the transform of the unit impulse function is equal to unity, then the inverse transformation of the product of the above-mentioned transforms leads to (6.21).

Substituting (6.21) in (6.20) we obtain:

$$u_{\text{out}}(0) = \int_0^{\infty} u_{\text{in}}(t) \omega_0 e^{-\alpha t} \sin \omega_0 t dt. \quad (6.22)$$

As had been shown already in § 21, the noise voltage at the output of a selective system is the result of superposition of infinitely many damped elementary oscillations, which are caused by elementary impulses of the input noise. A single elementary oscillation

$$du_{\text{out}} = u_{\text{in}}(t) dt \cdot \omega_0 e^{-\alpha t} \sin \omega_0 t \quad (6.23)$$

can be represented by a vector of length  $u_{\text{in}}(t) dt \cdot \omega_0 e^{-\alpha t}$  and of phase  $\omega_0 t$

(Figure 20). Expression (6.22) gives the projection of the resultant vector, representing the total oscillation, on the abscissa. As it is the length of the vector which is being considered, one has to write down the expression for its projection on the ordinate. This is equal to

$$v_{\text{out}}(0) = \int_0^{\infty} u_{\text{in}}(t) \omega_0 e^{-\alpha t} \cos \omega_0 t dt. \quad (6.24)$$

The projections  $u_{\text{out}}$  and  $v_{\text{out}}$  are random variables. If the input noise is uncorrelated, as will be assumed in the following, these projections are distributed normally. The normal law applies also to correlated external force having a normal distribution. The mean square of the projection  $u_{\text{out}}$  can be easily computed, using equation (3.132). Assuming in it  $\tau = 0$ , and noting that  $\alpha \ll \omega_0$ , we have:

$$\sigma_u^2 = S \int_0^{\infty} [\omega_0 e^{-\alpha t} \sin \omega_0 t]^2 dt \approx \frac{1}{4} \frac{\omega_0^2}{\alpha} S. \quad (6.25)$$

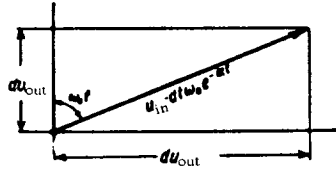


Figure 20. Vector of the elementary noise voltage at the output of a selective system

The same result is valid for the mean square of the projection of  $v_{\text{out}}$ . Considering the equations (6.22) and (6.24) as a set of two integral transformations of the random function, it is easy to calculate the mixed moment of the projections  $u$  and  $v$ . For this we make use of equation (3.153), in which we put  $t_2 = 0$ , denote  $t_1 - t_2 = t_1 = \tau$  and change the sign of the integration variable. Then this equation assumes the form:

$$m_2^{(uv)}(\tau) = S \int_0^{\infty} \xi_1(t + \tau) \xi_2(t) dt. \quad (6.26)$$

In the present case of  $\tau = 0$  the impulse characteristic  $\xi_1(t)$  is determined by equation (6.21), and  $\xi_2(t)$  is

$$\xi_2(t) = \omega_0 e^{-at} \cos \omega_0 t. \quad (6.27)$$

In view of the above, we have:

$$\begin{aligned} m_2^{(uv)}(0) &= S \int_0^\infty [\omega_0 e^{-at} \sin \omega_0 t] [\omega_0 e^{-at} \cos \omega_0 t] dt = \\ &= \frac{1}{2} S \omega_0^2 \int_0^\infty e^{-2at} \sin 2\omega_0 t dt = \frac{1}{4} S \frac{\omega_0^2}{a^2 + \omega_0^2}. \end{aligned} \quad (6.28)$$

From (6.25) and (6.28) we obtain the correlation coefficient between  $u$  and  $v$ :

$$\rho_{uv}(0) = \frac{m_2^{(uv)}(0)}{\sigma^2} = \frac{\omega_0/a}{1 + (\omega_0/a)^2}. \quad (6.29)$$

Since  $a \ll \omega_0$  we can assume

$$\rho_{uv}(0) = 0, \quad (6.30)$$

in other words, the projections can be considered as statistically independent of each other.

After this preliminary work we can proceed to the study of the distribution law of the amplitude  $U$  of the fluctuation oscillation, which is connected with the projections  $u$  and  $v$  by the nonlinear relationship

$$U = \sqrt{u^2 + v^2}. \quad (6.31)$$

For the solution of this problem we shall use the relations which were obtained in § 27 for an inertialess system with several inputs and outputs. The present case can be considered as having two inputs and one output, and it is required to find the one-dimensional distribution of the response. Equations (5.38) and (5.39) reduce to

$$w(y_{11}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A(z) e^{-iy_{11}z} dz, \quad (6.32)$$

with

$$A(z) = \int_{-\infty}^{+\infty} \exp[jz \cdot f(x_{11}, x_{21})] w(x_{11}, x_{21}) dx_{11} dx_{21}. \quad (6.33)$$

In our case we have  $x_{11} = u$ ,  $x_{21} = v$ ,  $y_{11} = U$ . The two-dimensional joint probability density for the projections  $u$  and  $v$  will be

$$w(u, v) = w(u) w(v) = \frac{1}{2\pi a^2} e^{-\frac{u^2 + v^2}{2a^2}}. \quad (6.34)$$

Substituting (6.31) and (6.34) in (6.33) we obtain:

$$A(z) = \frac{1}{2\pi a^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{u^2 + v^2}{2a^2}} e^{jz \sqrt{u^2 + v^2}} du dv. \quad (6.35)$$

Here the integration is performed over the entire  $uv$ -plane. We note that the integrand of (6.35) is constant for  $U = \sqrt{u^2 + v^2} = \text{const}$ . This allows to replace integration with respect to the coordinates  $u$  and  $v$  by radial integration, i. e., summation over annular bands of radius  $U$  and width  $dU$ , centered at the origin. Because of the constancy of the integrand, the integral with respect to a single ring is equal to

$$2\pi U dU e^{-\frac{U^2}{\sigma^2}} e^{jzU}. \quad (6.36)$$

In view of (6.36) we can express the characteristic function  $A(z)$  as follows:

$$A(z) = \int_0^{\infty} \frac{U}{\sigma^2} e^{-\frac{U^2}{\sigma^2}} e^{jzU} dU. \quad (6.37)$$

Combining (6.37) with the definition of the characteristic function

$$A(z) = \int_{-\infty}^{+\infty} w(x) e^{jzx} dx \quad (6.38)$$

and considering that the amplitude  $U$  can have only positive values, we conclude that the amplitude probability density is equal to

$$w(U) = \frac{U}{\sigma^2} e^{-\frac{U^2}{\sigma^2}}. \quad (6.39)$$

The same result is obtained by substituting the value (6.37) of the characteristic function in (6.32) and integrating. But in the present case we succeeded to carry out the calculation by a much simpler method.

The distribution law is expressed by (6.39) is called Rayleigh's law: the corresponding relation between the probability density  $w(U)$  and the amplitude is shown in Figure 21.



Figure 21. Probability density according to the Rayleigh distribution law

Let the analyzed noise voltage act on an ideal amplitude detector. We call an amplitude detector ideal if its transfer ratio is equal to unity and if it reproduces on its output without distortion the envelope of the modulated oscillation, acting on its input. The d.c. component of the voltage on the output of the detector, which is equal to the mathematical expectation of the amplitude  $U$ , is expressed as follows:

$$E_z = M[U] = \int_0^{\infty} U w(U) dU = \frac{1}{\sigma^2} \int_0^{\infty} U^2 e^{-\frac{U^2}{\sigma^2}} dU. \quad (6.40)$$

In tables of definite integrals /6/ we find the following formula:

$$\int_0^{\infty} e^{-px^2} x^{2a} dx = \frac{(2a-1)!!}{2(2p)^a} \sqrt{\frac{\pi}{p}}. \quad (6.41)$$

Using a particular case of this formula, with  $a = 1$ , we obtain:

$$E_u = \sqrt{\frac{\pi}{2}} \sigma. \quad (6.42)$$

Let us find the mean square of the amplitude  $U$ , which is at the same time the mean square of the voltage at the output of the ideal detector. We have:

$$\sigma_U^2 = \sigma_E^2 = \int_0^{\infty} U^2 w(U) dU = \frac{1}{\sigma^2} \int_0^{\infty} U^2 e^{-\frac{U^2}{2\sigma^2}} dU. \quad (6.43)$$

For the calculation of the integral (6.43) we shall use the following formula from the tables /6/:

$$\int_0^{\infty} e^{-px^2} x^{2a+1} dx = \frac{a!}{2p^{a+1}}. \quad (6.44)$$

In our case  $a = 1$ , and therefore

$$\sigma_U^2 = \sigma_E^2 = 2\sigma^2. \quad (6.45)$$

The mean square of the noise component of the voltage output from the detector is determined from the formula (6.42) and (6.45) as follows

$$\sigma_{E,n}^2 = \sigma_E^2 - E_u^2 = \frac{4-\pi}{2} \sigma^2 = 0.43\sigma^2. \quad (6.46)$$

Thus, all the more important characteristics of the random amplitude  $U$ , which could be calculated from its one-dimensional probability density have been obtained. Of course, the first and second moment (6.40) and (6.45) could have been found directly, without finding  $w(U)$ .

Let us consider now the statistical dependence in the envelope of the noise voltage. For this, we shall determine the two-dimensional probability density for the amplitudes  $U$  and  $U_\tau$ , separated by the time interval  $\tau$ . The relation between the amplitude  $U$  and its projections is shown in (6.31). Analogously, we have:

$$U_\tau = \sqrt{u_\tau^2 + v_\tau^2}. \quad (6.47)$$

As was shown above, the projections  $u$  and  $v$  in (6.31) are statistically independent. The same applies to the projections  $u_\tau$  and  $v_\tau$ . The statistical dependence between  $u$  and  $u_\tau$  is characterized by the moment (4.31), whence we obtain

$$\rho_{uu_\tau}(\tau) = e^{-\alpha\tau} \cos \omega_0 \tau. \quad (6.48)$$

We have to investigate the statistical dependences between the following pairs of projections:  $v$  and  $v_\tau$ ,  $u$  and  $v_\tau$ ,  $v$  and  $u_\tau$ . Applying (6.26) and (6.27) we can write for the projections  $v$  and  $v_\tau$ :

$$m_2^{(vv_\tau)}(\tau) = S \int_0^{\infty} [\omega_0 e^{-\alpha(t+\tau)} \cos \omega_0(t+\tau)] [\omega_0 e^{-\alpha t} \cos \omega_0 t] dt. \quad (6.49)$$

The calculation of the integral does not present any difficulties. Since  $\alpha \ll \omega_0$ , the result can be taken as equal to

$$m_2^{(vv)}(\tau) = \frac{1}{4} \frac{\omega_0^2}{\alpha} S e^{-\alpha\tau} \cos \omega_0 \tau, \quad (6.50)$$

whence we obtain, in view of (6.25) and (6.48):

$$\rho_{vv}(\tau) = \frac{m_2^{(vv)}(0)}{\sigma^2} = e^{-\alpha\tau} \cos \omega_0 \tau = \rho_{uu}(\tau). \quad (6.51)$$

Analogously, we have for the projections  $u$  and  $v$ :

$$\begin{aligned} m_2^{(uv)}(\tau) &= S \int_0^\infty [\omega_0 e^{-\alpha(t+\tau)} \sin \omega_0(t+\tau)] [\omega_0 e^{-\alpha t} \cos \omega_0 t] dt = \\ &= \frac{1}{4} \frac{\omega_0^2}{\alpha} S e^{-\alpha\tau} \sin \omega_0 \tau, \end{aligned} \quad (6.52)$$

i.e.,

$$\rho_{uv}(\tau) = e^{-\alpha\tau} \sin \omega_0 \tau, \quad (6.53)$$

and for the projections  $v$  and  $u$ :

$$\begin{aligned} m_2^{(vu)}(\tau) &= S \int_0^\infty [\omega_0 e^{-\alpha(t+\tau)} \cos \omega_0(t+\tau)] [\omega_0 e^{-\alpha t} \sin \omega_0 t] dt = \\ &= -\frac{1}{4} \frac{\omega_0^2}{\alpha} S e^{-\alpha\tau} \sin \omega_0 \tau, \end{aligned} \quad (6.54)$$

whence

$$\rho_{vu}(\tau) = -e^{-\alpha\tau} \sin \omega_0 \tau. \quad (6.55)$$

Henceforth it will be convenient to use the following notations for the expressions (6.48), (6.51), (6.53) and (6.55):

$$\rho_{uu} = \rho_{vv} = \psi(\tau) \cos \omega_0 \tau = \psi(\tau) \cos \beta, \quad (6.56)$$

$$\rho_{uv} = -\rho_{vu} = \psi(\tau) \sin \omega_0 \tau = \psi(\tau) \sin \beta, \quad (6.57)$$

where  $\psi(\tau)$  is the envelope of the normalized correlation function of the instantaneous noise values, which for a single oscillation circuit is equal to

$$\psi(\tau) = e^{-\alpha\tau} \quad (6.58)$$

Now we can write down the four-dimensional probability density of the projections. As well known, with a normal distribution law the  $n$ -dimensional probability density can be expressed as follows [40]:

$$\begin{aligned} w(x_1, x_2, \dots, x_n) &= \\ &= \frac{1}{\sigma_1 \sigma_2 \dots \sigma_n \sqrt{(2\pi)^n D}} \exp \left[ -\frac{1}{2D} \sum_{i,j=1}^n D_{ij} \frac{x_i x_j}{\sigma_i \sigma_j} \right], \end{aligned} \quad (6.59)$$

where  $D$  is the determinant of  $n$ -th order:

$$D = \begin{vmatrix} \rho_{11} & \rho_{12} & \dots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \dots & \rho_{2n} \\ \dots & \dots & \dots & \dots \\ \rho_{n1} & \rho_{n2} & \dots & \rho_{nn} \end{vmatrix}. \quad (6.60)$$

$D_{ij}$  is the algebraic complement of the element  $\rho_{ij}$  of this determinant,  $\rho_{ij}$  being the correlation coefficient. For  $i = j$  we have  $\rho_{ij} = 1$ . In the present case, putting

$x_1 = u$ ,  $x_2 = v$ ,  $x_3 = u_r$ , and  $x_4 = v_r$ , we have:

$$D = \begin{vmatrix} 1 & 0 & \psi \cos \beta & \psi \sin \beta \\ 0 & 1 & -\psi \sin \beta & \psi \cos \beta \\ \psi \cos \beta & -\psi \sin \beta & 1 & 0 \\ \psi \sin \beta & \psi \cos \beta & 0 & 1 \end{vmatrix}. \quad (6.61)$$

The calculation of  $D$  and of its algebraic complements  $D_{ij}$  gives the following results:

$$D = (1 - \psi^2)^2, \quad (6.62)$$

$$D_{11} = D_{22} = D_{33} = D_{44} = 1 - \psi^2, \quad (6.63)$$

$$D_{12} = D_{21} = D_{34} = D_{43} = 0, \quad (6.64)$$

$$D_{13} = D_{31} = D_{24} = D_{42} = -\psi(1 - \psi^2) \cos \beta, \quad (6.65)$$

$$D_{14} = D_{41} = -D_{23} = -D_{32} = -\psi(1 - \psi^2) \sin \beta. \quad (6.66)$$

With these results the general expression (6.59) becomes:

$$w(u, v, u_r, v_r) = \frac{1}{4\pi^2 \sigma^4 (1 - \psi^2)} \exp \left[ -\frac{1}{2\sigma^2 (1 - \psi^2)} (u^2 + v^2 + u_r^2 + v_r^2 - 2\psi [\cos \beta (uu_r + vv_r) + \sin \beta (uv_r - vu_r)]) \right]. \quad (6.67)$$

The characteristic function of the probability density of the amplitudes  $U$  and  $U_r$  can be expressed thus:

$$A(z_1, z_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp [j \{z_1 \sqrt{u^2 + v^2} + z_2 \sqrt{u_r^2 + v_r^2}\}] \times \\ \times w(u, v, u_r, v_r) du dv du_r dv_r. \quad (6.68)$$

by analogy with expression (5.35).

Let us introduce new integration variables, which are connected with those of (6.68) by the relations

$$u = U \sin \theta, \quad v = U \cos \theta, \quad u_r = U_r \sin \theta_r, \quad v_r = U_r \cos \theta_r. \quad (6.69)$$

Then, using the rule for change of variables in multiple integrals, we give to the characteristic function (6.68) the form

$$A(z_1, z_2) = \int_0^{+\infty} \int_0^{+\infty} \exp [j(z_1 U + z_2 U_r)] \frac{UU_r}{4\pi^2 \sigma^4 (1 - \psi^2)} \times \\ \times \exp \left[ -\frac{1}{2\sigma^2 (1 - \psi^2)} (U^2 + U_r^2) \right] dU dU_r \times \\ \times \int_0^{2\pi} \int_0^{2\pi} \exp \left[ \frac{jUU_r}{\sigma^2 (1 - \psi^2)} \cos(\theta - \theta_r - \beta) \right] d\theta d\theta_r. \quad (6.70)$$

The calculation of the inner integral is simple, in view of formula

$$\int_0^{2\pi} e^{a \cos(x-b)} dx = 2\pi I_0(a). \quad (6.71)$$

This formula is easily obtained by replacing the exponential function in the integrand by a power series, by integrating by parts and comparing the obtained result with the power series (6.15). Using expression (6.71), we obtain:

$$\int_0^{2\pi} \int_0^{2\pi} \exp \left[ \frac{\psi U U_\tau}{\sigma^2 (1 - \psi^2)} \cos(\theta - \theta_\tau - \beta) \right] d\theta d\theta_\tau = \\ = 4\pi^2 I_0 \left[ \frac{\psi U U_\tau}{\sigma^2 (1 - \psi^2)} \right], \quad (6.72)$$

which allows to write (6.70) in the form

$$A(z_1, z_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp [j(z_1 U + z_2 U_\tau)] \frac{U U_\tau}{\sigma^4 (1 - \psi^2)} \times \\ \times I_0 \left[ \frac{\psi U U_\tau}{\sigma^2 (1 - \psi^2)} \right] \exp \left[ -\frac{U^2 + U_\tau^2}{2\sigma^2 (1 - \psi^2)} \right] dU dU_\tau. \quad (6.73)$$

Comparing (6.73) with the definition of the characteristic two-dimensional distribution function

$$A(z_1, z_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp [j(z_1 U + z_2 U_\tau)] w(U, U_\tau) dU dU_\tau, \quad (6.74)$$

we obtain the following final result:

$$w(U, U_\tau) = \frac{U U_\tau}{\sigma^4 (1 - \psi^2)} I_0 \left[ \frac{\psi U U_\tau}{\sigma^2 (1 - \psi^2)} \right] \exp \left[ -\frac{U^2 + U_\tau^2}{2\sigma^2 (1 - \psi^2)} \right]. \quad (6.75)$$

If the time interval  $\tau$  is very long ( $\tau \rightarrow \infty$ ), then  $\psi(\tau) \rightarrow 0$  and with  $I_0(0) = 1$  we have

$$w(U, U_\tau) = \frac{U}{\sigma^2} e^{-\frac{U^2}{2\sigma^2}} \cdot \frac{U_\tau}{\sigma^2} e^{-\frac{U_\tau^2}{2\sigma^2}}, \quad (6.76)$$

i.e., (6.75) becomes a product of one-dimensional probability densities of the kind (6.39). This was to be expected, as at  $\tau \rightarrow \infty$  the amplitudes  $U$  and  $U_\tau$  become statistically independent of each other.

The mixed moment of amplitudes  $U$  and  $U_\tau$  can be found in the following way:

$$m_2^{(U)}(\tau) = \int_0^\infty \int_0^\infty U U_\tau w(U, U_\tau) dU dU_\tau, \quad (6.77)$$

where the probability density  $w(U, U_\tau)$  is expressed by formula (6.75). Thus,

$$m_2^{(U)}(\tau) = \int_0^\infty \int_0^\infty \frac{U^2 U_\tau^2}{\sigma^4 (1 - \psi^2)} I_0 \left[ \frac{\psi U U_\tau}{\sigma^2 (1 - \psi^2)} \right] \times \\ \times \exp \left[ -\frac{U^2 + U_\tau^2}{2\sigma^2 (1 - \psi^2)} \right] dU dU_\tau. \quad (6.78)$$

The calculation of the integral (6.78) leads to rather tedious computations. They are contained in the monograph by V.I. Bunimovich /4/ and will not be repeated here. We give only the final result:

$$m_2^{(U)}(\tau) = \sigma^2 [2E(\psi) - (1 - \psi^2) K(\psi)], \quad (6.79)$$

where  $K(\psi)$  and  $E(\psi)$  are complete elliptic integrals of the first and second class,

$$K(\psi) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \psi^2 \sin^2 \varphi}}, \quad (6.80)$$

$$E(\psi) = \int_0^{\pi/2} \sqrt{1 - \psi^2 \sin^2 \varphi} d\varphi. \quad (6.81)$$

The mixed moment for the fluctuation component of the amplitude and, simultaneously, of the voltage at the output of the ideal detector becomes, by virtue of (6.42) and (6.79),

$$m_{2,n}^{(U)}(\tau) = m_2^{(U)}(\tau) - E_-^2 = \sigma^2 \left\{ [2E(\psi) - (1 - \psi^2)K(\psi)] - \frac{\pi}{2} \right\}. \quad (6.82)$$

As complete elliptic integrals can be expanded into rapidly converging series in terms of even powers of  $\psi$ , we can expect that expression (6.82) can be approximated with sufficient accuracy by a polynomial with a small number of terms. Indeed, the expression (6.82) can be very well replaced by the approximate formula

$$m_{2,n}^{(U)}(\tau) \approx \sigma^2 [0.39\psi^2(\tau) + 0.04\psi^4(\tau)]. \quad (6.83)$$

For tentative calculations the still rougher approximation

$$m_{2,n}^{(U)}(\tau) \approx 0.43\sigma^2\psi^2(\tau). \quad (6.84)$$

can be used.

Results of calculation by the exact formula (6.82) and by the two approximations (6.83) and (6.84) show that the error introduced by the formula (6.83) does not exceed 1%, and can reach 10% with the formula (6.84).

In conclusion, let us examine the spectrum of the envelope or of the voltage at the output of an ideal detector. Using the approximate formula (6.84) and assuming that the envelope  $\psi(\tau)$  of the normalized correlation function for the instantaneous noise values satisfies the relation (6.58), as it does in the case of a single oscillation circuit, we obtain from (3.62):

$$F_U(\omega) = \frac{2}{\pi} \int_0^\infty 0.43\sigma^2 e^{-2\alpha\tau} \cos \omega\tau d\tau = 0.55\sigma^2 \frac{\alpha}{4\alpha^2 + \omega^2}. \quad (6.85)$$

For definiteness, the above analysis has been restricted to noise voltage on a single oscillation circuit. But it can be easily seen that this assumption determines only the form of the function  $\psi(\tau)$ . The rest of the obtained results can be applied to any selective system.

In the theory of radio reception, in addition to the above-considered problem, it is also of interest to examine the statistical properties of the envelope when fluctuation noise and signal of some shape are simultaneously applied to the input of the selective system. In such a case the analysis is accomplished by the same means, but the computation turns out to be considerably more cumbersome.

### §31. Statistical Properties of the Phase of a Noise Voltage at the Output of a Selective System

Phase fluctuations of a noise voltage at the output of a selective system are

of interest in the case when this noise acts on some phase-sensitive device. The following final results have been taken from V.I. Bunimovich /4/.

We understand by the random phase of the noise voltage the angle formed by the vector  $U$ , which represents the random process, and the abscissa. This random phase was already introduced in the foregoing section, (the angles  $\theta$  and  $\theta_1$  in the expressions (6.69) and the following). From the first pair of the relations (6.69) we find that the nonlinear transformation to be examined is, in the present case, of the form

$$\theta = \arctg \frac{u}{v}. \quad (6.86)$$

Starting from (6.33) and (6.34) we shall write down the characteristic function for the probability density of the phase as follows:

$$A(z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp \left[ jz \arctg \frac{u}{v} \right] \cdot \frac{1}{2\pi\sigma^2} e^{-\frac{u^2+v^2}{2\sigma^2}} du dv. \quad (6.87)$$

Replacing in (6.87) the integration variables  $u$  and  $v$  by the new variables  $U$  and  $\theta$ , we obtain

$$A(z) = \int_0^{2\pi} e^{-jz\theta} d\theta \cdot \frac{1}{2\pi\sigma^2} \int_0^{\infty} U e^{-\frac{U^2}{2\sigma^2}} dU. \quad (6.88)$$

Applying formula (6.44) taken from the tables, for  $a = 0$  we can transform the equation (6.88) to

$$A(z) = \int_0^{2\pi} e^{jz\theta} \frac{d\theta}{2\pi}. \quad (6.89)$$

Now, combining (6.89) with the definition (6.38) of the characteristic function of the one-dimensional distribution, we obtain

$$w(\theta) = \frac{1}{2\pi}. \quad (6.90)$$

Thus, all values of the phase between zero and  $2\pi$  are equally probable.

We turn now to the two-dimensional distribution of the phase. The corresponding characteristic function can be written down, by analogy with the equation (5.35), in the following form

$$A(z_1, z_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp \left[ j \left\{ z_1 \arctg \frac{u}{v} + z_2 \arctg \frac{u_1}{v_1} \right\} \right] \times \\ \times w(u, v, u_1, v_1) du dv du_1 dv_1, \quad (6.91)$$

where the four-dimensional probability density  $w(u, v, u_1, v_1)$  is determined by formula (6.67). By changing in (6.91) the integration variables in accordance with (6.69) we obtain

$$A(z_1, z_2) = \\ = \int_0^{2\pi} \int_0^{2\pi} \exp [j(z_1\theta + z_2\theta_1)] d\theta d\theta_1 \int_0^{\infty} \int_0^{\infty} \frac{UU_1}{4\pi^2\sigma^4(1-\psi^2)} \times \\ \times \exp \left[ -\frac{U^2 - 2\psi \cos(\theta - \theta_1 - \beta) UU_1 + U_1^2}{2\sigma^2(1-\psi^2)} \right] dU dU_1. \quad (6.92)$$

In the paper of S. Rice /39/ the following definite integral was computed:

$$\int_0^\infty \int_0^\infty xy \cdot \exp [-(x^2 + 2xy \cos \varphi + y^2)] dx dy = \frac{1}{4} \operatorname{cosec} \varphi (1 - \tau \operatorname{ctg} \varphi). \quad (6.93)$$

If, in the inner integral of (6.92), the variables are changed

$$\frac{U^2}{2\sigma^2(1-\psi^2)} = x^2, \quad \frac{U_\tau^2}{2\sigma^2(1-\psi^2)} = y^2 \quad (6.94)$$

and

$$\psi \cos(\theta - \theta_\tau - \beta) = \eta = -\cos \varphi. \quad (6.95)$$

is introduced, then its computation amounts to the use of formula (6.93). In view of this the expression (6.92) assumes by simple transformations the following form:

$$A(z_1, z_2) = \int_0^{2\pi} \int_0^{2\pi} \exp [j(z_1 \theta + z_2 \theta_\tau)] d\theta d\theta_\tau \times \\ \times \frac{1-\psi^2}{4\pi^2} \left\{ \frac{1}{1-\eta^2} + \eta \frac{\pi - \arccos \eta}{(1-\eta^2)^{3/2}} \right\}. \quad (6.96)$$

Combining (6.96) with (6.74) we obtain the two-dimensional probability density of the phase:

$$w(\theta, \theta_\tau) = \frac{1-\psi^2}{4\pi^2} \left\{ \frac{1}{1-\eta^2} + \eta \frac{\pi - \arccos \eta}{(1-\eta^2)^{3/2}} \right\}, \quad (6.97)$$

where the quantity  $\eta$  is determined by equation (6.95). We recall that in the considerations of the foregoing section, on which the calculations preceding (6.97) were based, it was assumed that  $t_1 > t_2$ . In other words, the phase  $\theta$  corresponds to a time moment which follows the phase  $\theta_\tau$  after an interval  $\tau$ .

If it is assumed that  $\tau \rightarrow \infty$ , then  $\psi(\tau) \rightarrow 0$ ,  $\eta \rightarrow 0$  and the probability density (6.97) at the limit assumes the value

$$w(\theta, \theta_\tau) = \frac{1}{2\pi} \cdot \frac{1}{2\pi}. \quad (6.98)$$

Comparison of (6.98) with the one-dimensional probability density (6.90) shows that at  $\tau \rightarrow \infty$  both phases become statistically independent of each other, as was to be expected.

Let us consider the statistical properties of the phase  $\theta_\tau$  when the value has been fixed at some time moment, which is being taken as the beginning of the reading. For this, we must find for  $\theta$  the conditional probability:

$$w(\theta_\tau | \theta) = \frac{w(\theta, \theta_\tau)}{w(\theta)}. \quad (6.99)$$

In view of expressions (6.90) and (6.97) for the two probability densities entering in (6.99) we obtain:

$$w(\theta_\tau | \theta) = \frac{1-\psi^2}{2\pi} \left[ \frac{1}{1-\eta^2} + \eta \frac{\pi - \arccos \eta}{(1-\eta^2)^{3/2}} \right]. \quad (6.100)$$

As follows from (6.95) the quantity  $\eta$  satisfies the inequality  $0 \leq \eta \leq 1$ . It is easily seen that the probability density  $w(\theta, |\theta)$  increases monotonically and boundlessly as  $\eta$  approaches unity.

At fixed interval  $\tau$  the maximum of the quantity  $\eta$  and, therefore, of the conditional probability  $w(\theta, |\theta)$  occurs at  $\cos(\theta - \theta_0 - \beta) = 1$ , which corresponds to the condition

$$\theta = \theta_0 + \beta = \theta_0 + \omega_0 \tau. \quad (6.101)$$

Expression (6.101) gives the most probable value of the phase  $\theta$ , if at the time moment taken as the beginning of reading the phase were  $\theta_0$ . From the right-hand side of (6.101) a multiple of  $2\pi$  was excluded, since, at  $\tau = 0$ , it is natural to require that  $\theta = \theta_0$ .

As the cosine is an even function, deviation of the phase  $\theta$  from its most probable value (6.101) causes a decrease of the probability density  $w(\theta, |\theta)$ , which is independent of the sign of this deviation. Thus, the conditional distribution of the phase  $\theta$  has an axis of symmetry, in a position defined by (6.101).

As well known, in any distribution which has an axis of symmetry the most probable value of the random variable coincides with its mathematical expectation. Therefore, expression (6.101) gives at the same time the mathematical expectation of the phase at the moment  $\tau$ , whence follows that the vector representing the random process rotates in the mean with the angular velocity  $\omega_0$ . This conclusion agrees fully with the notion of the noise voltage at the output of a selective system being an oscillation of frequency  $\omega_0$  with amplitude and phase changing at random.

We find the height of the maximum of the conditional probability density from (6.100) by substituting  $\eta = \psi$ :

$$w_{\max}(\theta, |\theta) = \frac{1}{2\pi} \left[ 1 + \psi \frac{\pi - \arccos \psi}{\sqrt{1 - \psi^2}} \right]. \quad (6.102)$$

At small values of  $\tau$  the envelope  $\psi(\tau)$  of the correlation function of the instantaneous noise voltage is near to unity and the probability density  $w_{\max}(\theta, |\theta)$  is large. This means that the possible values of the phase  $\theta$  are mainly concentrated near the most probable value (6.101), i.e., the dispersion of the phase is insignificant. As  $\tau$  increases the height of the maximum drops and the distribution becomes diffuse—the dispersion grows. At  $\tau \rightarrow \infty$  we have  $\psi(\tau) \rightarrow 0$  and the distribution of the phase becomes uniform:

$$w_{\max}(\theta, |\theta) = w(\theta) = \frac{1}{2\pi}. \quad (6.103)$$

It was assumed above that the phase  $\theta$  and  $\theta_0$  are random variables with possible values between zero and  $2\pi$ . In the expression (6.100) for the conditional probability this corresponds to fluctuations of the phase  $\theta$  about its mean value within the limits of  $-\pi$  and  $+\pi$ . Since the selective system is completely indifferent to the phases of the oscillations occurring in it, the phase can deviate arbitrarily far and with any probability from its mathematical expectation. The applied method of analysis is such that when the phase passes the limits  $\pm\pi$  the terms  $\pm\pi$  are automatically dropped; if this is not done the phase strays from its mean value similarly to the Brownian particle considered in § 19.

§32. Statistical Dependence Between Envelopes of Noise Voltages  
at the Outputs of Two Selective Four-poles with their Inputs  
Connected in Parallel

The solution of this question is of interest in cases where the outputs of two selective four-poles, with their inputs connected in parallel, are connected to detectors whose output voltages are added together in some way. Such systems are often encountered in FM discriminators of radio receivers, which are used for automatic frequency control of the heterodyne and also for radio reception of frequency-modulated oscillations. The rectification of voltages from the outputs of two selective four-poles and their subsequent addition is also encountered in diversity reception, where, to overcome fading, the reception is simultaneously done by two receivers, tuned to different frequencies, whose signals are added together after detection.

In order to avoid cumbersome calculations only a simple particular case will be considered below, in which the four-poles, tuned to the frequencies  $\omega_1$  and  $\omega_2$  respectively, have similar transfer characteristics of the form shown in Figure 22. We recall that by the transfer characteristic of a selective system we mean the law of steady state of the amplitude of its output voltage upon instantaneously switching on a sinusoidal input voltage of unit amplitude.

The envelope of the impulse characteristic is the time derivative of the transfer characteristic and has in our case a rectangular form (Figure 23). The high-frequency complement of the impulse characteristic has the same frequency as the natural frequency of the selective system, as was already shown in §30. The phase of this complement is not essential for the following. We can write, therefore, the equations for the impulse characteristics of four-poles as follows:

$$\xi_1(t) = \begin{cases} \frac{K_0}{a} \sin \omega_1 t & (0 \leq t \leq a), \\ 0 & (t > a), \end{cases} \quad (6.104)$$

$$\xi_2(t) = \begin{cases} \frac{K_0}{a} \sin \omega_2 t & (0 \leq t \leq a), \\ 0 & (t > a). \end{cases} \quad (6.105)$$

The noise voltage at the common input of the four-poles will be supposed to be uncorrelated. We shall analyze the statistical dependence between the envelopes at one and the same time moment. This allows to compute the mean square of the noise after addition at the outputs. If the spectrum of the resulting noise is also of interest, then one has to examine also the statistical dependence between values of envelopes separated by some interval  $\tau$ . Such a generalization does not present substantial difficulties.

The projections on the coordinates of the vectors representing noise voltages at the outputs of the four-poles are by analogy to (6.22) and (6.24):

$$u_1 = \int_0^a u_{in}(t) \frac{K_0}{a} \sin \omega_1 t dt, \quad (6.106)$$

$$v_1 = \int_0^a u_{in}(t) \frac{K_0}{a} \cos \omega_1 t dt, \quad (6.107)$$

$$u_2 = \int_0^a u_{in}(t) \frac{K_0}{a} \sin \omega_2 t dt, \quad (6.108)$$

$$v_2 = \int_0^a u_{in}(t) \frac{K_0}{a} \cos \omega_2 t dt. \quad (6.109)$$

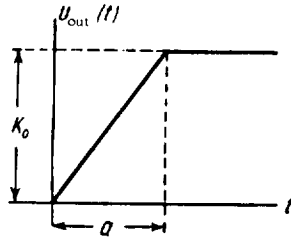


Figure 22. Transfer characteristic of a single selective four-pole

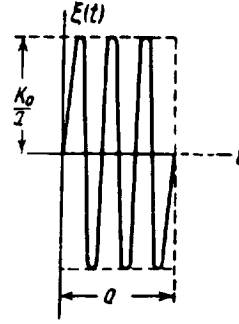


Figure 23. Impulse characteristic of a single selective four-pole

As already established in §30, the random variables  $u_1$  and  $v_1$  can be considered as statistically independent. The same applies to the variables  $u_2$  and  $v_2$ .

We calculate the mixed moment for the projections  $u_1$  and  $u_2$  with the help of equation (6.26), assuming in it  $\tau = 0$  and considering that  $|\omega_1 - \omega_2| \ll \omega_1 + \omega_2$ :

$$m_2^{(u, u_2)} = S \int_0^a \frac{K_0^2}{a^2} \sin \omega_1 t \sin \omega_2 t dt \approx \frac{1}{2} S \frac{K_0^2}{a^2} \frac{\sin \Delta \omega a}{\Delta \omega}, \quad (6.110)$$

where  $\Delta \omega = \omega_1 - \omega_2$ . If we put  $\Delta \omega = 0$  then both four-poles become identical, and (6.110) gives the value of the mean square of the noise at the output of each.

$$m_2^{(u, u_1)} = \sigma^2 = \frac{1}{2} S \frac{K_0^2}{a^2}. \quad (6.111)$$

The correlation coefficient between the projections  $u_1$  and  $u_2$  is equal to

$$\rho_{u, u_2} = \frac{\sin \Delta \omega a}{\Delta \omega a}. \quad (6.112)$$

Analogously, we find:

$$m_2^{(v, v_2)} = S \int_0^a \frac{K_0^2}{a^2} \cos \omega_1 t \cos \omega_2 t dt \approx \frac{1}{2} S \frac{K_0^2}{a^2} \frac{\sin \Delta \omega a}{\Delta \omega}, \quad (6.113)$$

i. e.,

$$\rho_{v, v_2} = \rho_{u, u_2}. \quad (6.114)$$

For the projections  $u_1$  and  $v_2$  we have:

$$m_2^{(u, v_1)} = S \int_0^{\pi} \frac{K_0^2}{a^2} \sin \omega_1 t \cos \omega_2 t dt \approx -\frac{1}{2} S \frac{K_0^2}{a^2} \frac{1 - \cos \Delta \omega a}{\Delta \omega}, \quad (6.115)$$

whence we obtain:

$$\rho_{u, v_1} = -\frac{1 - \cos \Delta \omega a}{\Delta \omega a}. \quad (6.116)$$

The expression for the correlation coefficient between the projections  $u_2$  and  $v_1$  has the form:

$$\rho_{u, v_1} = \frac{1 - \cos \Delta \omega a}{\Delta \omega a}. \quad (6.117)$$

After having found all correlation coefficients between all projections we have to make the calculations analogous to those of §30. The determinant D (6.60) has in the present case the form

$$D = \begin{vmatrix} 1 & 0 & \frac{\sin x}{x} & -\frac{1 - \cos x}{x} \\ 0 & 1 & \frac{1 - \cos x}{x} & \frac{\sin x}{x} \\ \frac{\sin x}{x} & \frac{1 - \cos x}{x} & 1 & 0 \\ -\frac{1 - \cos x}{x} & \frac{\sin x}{x} & 0 & 1 \end{vmatrix}, \quad (6.118)$$

where  $x = \Delta \omega a$ . The computation of the determinant (6.118) gives

$$D = \left[ 1 - \frac{2}{x^2} (1 - \cos x) \right]^2. \quad (6.119)$$

The algebraic complements of the determinant (6.118) are

$$D_{11} = D_{22} = D_{33} = D_{44} = 1 - \frac{2}{x^2} (1 - \cos x), \quad (6.120)$$

$$D_{12} = D_{21} = D_{34} = D_{43} = 0, \quad (6.121)$$

$$D_{13} = D_{31} = D_{24} = D_{42} = -\frac{\sin x}{x} \left[ 1 - \frac{2}{x^2} (1 - \cos x) \right], \quad (6.122)$$

$$D_{14} = D_{41} = -D_{23} = -D_{32} = -\frac{1 - \cos x}{x} \left[ 1 - \frac{2}{x^2} (1 - \cos x) \right]. \quad (6.123)$$

Using equalities (6.119) to (6.123), we can write down the four-dimensional probability density of the projections, in accordance with (6.59) as follows:

$$\begin{aligned} w(u_1, v_1, u_2, v_2) &= \frac{1}{4\pi^2 a^4 \left[ 1 - \frac{2}{x^2} (1 - \cos x) \right]} \times \\ &\times \exp \left[ -\frac{1}{2a^2 \left[ 1 - \frac{2}{x^2} (1 - \cos x) \right]} \left\{ u_1^2 + v_1^2 + u_2^2 + v_2^2 - \right. \right. \\ &\left. \left. - 2 \left[ \frac{\sin x}{x} (u_1 u_2 + v_1 v_2) + \frac{1 - \cos x}{x} (u_1 v_2 - v_1 u_2) \right] \right\} \right]. \end{aligned} \quad (6.124)$$

The characteristic function of the two-dimensional probability density for the amplitudes  $U_1$  and  $U_2$  is expressed analogously to (6.68). By introducing new variables

$$u_1 = U_1 \sin \theta_1, \quad v_1 = U_1 \cos \theta_1, \quad u_2 = U_2 \sin \theta_2, \quad v_2 = U_2 \cos \theta_2, \quad (6.125)$$

we obtain

$$\begin{aligned}
A(z_1, z_2) = & \int_0^\infty \int_0^\infty \exp \{ j(z_1 U_1 + z_2 U_2) \} \frac{U_1 U_2}{4\pi^2 a^4 (1 - \psi^2)} \times \\
& \times \exp \left[ -\frac{1}{2a^2 (1 - \psi^2)} (U_1^2 + U_2^2) \right] dU_1 dU_2 \times \\
& \times \int_0^{2\pi} \int_0^{2\pi} \exp \left[ \frac{j U_1 U_2}{a^2 (1 - \psi^2)} \cos(\theta_1 - \theta_2 - \beta) \right] d\theta_1 d\theta_2,
\end{aligned} \quad (6.126)$$

where

$$\psi = \frac{\sin(x/2)}{x/2} = \frac{\sin(\Delta\omega a/2)}{\Delta\omega a/2}, \quad (6.127)$$

$$\beta = \arctg \frac{1 - \cos x}{\sin x}. \quad (6.128)$$

The characteristic function (6.126) has a completely analogous form to the expression (6.70). This allows to write down without further calculations the two-dimensional probability density for the amplitudes  $U_1$  and  $U_2$ , by analogy with (6.75):

$$w(U_1, U_2) = \frac{U_1 U_2}{a^4 (1 - \psi^2)} J_0 \left[ \frac{\psi U_1 U_2}{a^2 (1 - \psi^2)} \right] \exp \left[ -\frac{U_1^2 + U_2^2}{2a^2 (1 - \psi^2)} \right]. \quad (6.129)$$

As in (6.75), the obtained formula shows that at  $\psi \rightarrow 0$  the two amplitudes become statistically independent of each other. In the present case, it follows from (6.127) that this occurs at  $\Delta\omega \rightarrow \infty$  or at  $a \rightarrow \infty$ , i.e., either for unlimited increase of the difference between the two frequencies  $\omega_1$  and  $\omega_2$  or for unlimited contraction of the frequency bands passed by both four-poles (as known, the pass band width of a four-pole is inversely proportional to the settling time  $a$ ).

The identity of the expressions (6.75) and (6.129) allows to use for the computation of the mixed moment of the amplitudes  $U_1$  and  $U_2$  the exact formula (6.79) as well as the approximations (6.83) and (6.84). It follows from the latter, from (6.46) and from (6.127), that the correlation coefficient between the amplitudes  $U_1$  and  $U_2$  is approximately equal to

$$\rho_{12} \approx \left[ \frac{\sin(\Delta\omega a/2)}{\Delta\omega a/2} \right]^2. \quad (6.130)$$

Let us examine the change in the correlation coefficient with increased detuning between the four-poles. For this purpose we shall examine the graph of Figure 24. It is seen that the correlation between the envelopes of the noise voltages vanishes practically ( $\rho_{12} \leq 0.05$ ) at  $\Delta\omega a \geq \frac{5}{3} \pi$ , i.e., at

$$a \cdot \Delta f \geq \frac{5}{6} = 0.83. \quad (6.131)$$

If it is taken into account that the settling time and the pass band  $\Delta F$  of a selective system are connected by the approximate relation

$$a \cdot \Delta F \approx 1, \quad (6.132)$$

it is easily seen that condition (6.131) is to be spectrally interpreted as a detuning in which the two four-poles block non-overlapping bands from the input noise spectrum.

If the two four-poles are terminated by ideal detectors (in the sense of § 30) whose output voltages are connected in opposition (according to the polarity of the d. c. components of these voltages) then the mean square of the resulting noise voltage becomes, in accordance with (6.46) and (6.131):

$$\sigma_{\text{res}}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2 = 0.86\sigma^2 \left\{ 1 - \left[ \frac{\sin\left(\frac{\Delta\omega a}{2}\right)}{\frac{\Delta\omega a}{2}} \right]^2 \right\}. \quad (6.133)$$

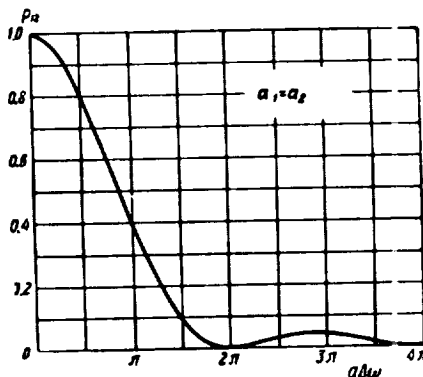


Figure 24. Graph of the normalized correlation function for noise voltages at outputs of four-poles

This expression shows that the mean square of the resulting noise varies as a function of the product  $\Delta\omega a$  between zero and  $0.86\sigma^2$ .

### §33. Statistical Properties of the Voltage Envelope at the Output of a Selective System Under Action of Nonmodulated Signal and Fluctuation Noise

We return to the problem considered in §30 and add to it the following condition. We assume that the selective system is acted upon by the fluctuation noise as well as by a nonmodulated oscillation of amplitude  $U_0$ , and of frequency equal to the resonance frequency of the generator. This case was also analyzed in the publications cited in the above-mentioned section.

As far as the instantaneous values of the voltages and currents are concerned the system is linear, and, by the principle of superposition, the random and the regular processes, which take place in the given case, can be analyzed independently. But, as had been already shown in §30, the problem of the envelope is nonlinear, and therefore the nonmodulated oscillation signal and the fluctuation noise must be considered together.

Let us find the one-dimensional distribution of the envelope of the output voltage. As was shown in §31, all phases of the noise voltage vector are equally probable. In other words, the projection plane  $uv$  has complete radial symmetry. This allows to rotate arbitrarily the axes of  $u$  and  $v$ , preserving, of course, their mutual perpendicularity. For the present problem it is convenient to orientate the  $u$  axis along the vector  $U_0$  of the signal and the  $v$  axis perpendicular to it. Then the length of the resulting voltage vector can be written as

$$U = \sqrt{(U_0 + u)^2 + v^2}. \quad (6.134)$$

Applying formula (6.34) for the two-dimensional probability density of the projections  $u$  and  $v$  and the general expression (6.33) for the characteristic function, we obtain:

$$A(z) = \int_{-\infty}^{+\infty} \int e^{jz \sqrt{(U_0+u)^2+v^2}} \frac{1}{2\pi\sigma^2} e^{-\frac{u^2+v^2}{2\sigma^2}} du dv. \quad (6.135)$$

We introduce new integration variables  $U$  and  $\theta$ , connected with  $u$  and  $v$  by the relations

$$U_0 + u = U \cos \theta; \quad v = U \sin \theta. \quad (6.136)$$

Then the characteristic function (6.135) becomes

$$A(z) = \int_0^\infty e^{jzU} \frac{U}{2\pi\sigma^2} e^{-\frac{U^2+U_0^2}{2\sigma^2}} dU \int_0^{2\pi} e^{\frac{U U_0}{\sigma^2} \cos \theta} d\theta. \quad (6.137)$$

By calculating with the aid of the formula (6.71) the inner integral of (6.137) and combining the obtained result with (6.38) we obtain:

$$w(U) = \frac{U}{\sigma^2} I_0 \left( \frac{U U_0}{\sigma^2} \right) e^{-\frac{U^2+U_0^2}{2\sigma^2}}. \quad (6.138)$$

In the absence of a signal ( $U_0 = 0$ ) and since  $I_0(0) = 1$ , the expression (6.138) transforms, as expected, into (6.39). The other extreme is the case of very strong signals. It is clear from geometrical considerations that for very large values of  $U_0$  it is not essential to account for the projection  $v$ . It follows, that in this case the difference  $U - U_0$  coincides with the projection  $u$ , and therefore

$$w(U) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(U-U_0)^2}{2\sigma^2}}. \quad (6.139)$$

This result can be also readily obtained from (6.138) by the use of the asymptotical expression for the Bessel function

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}. \quad (6.140)$$

Let us find the mean value of the amplitude  $U$ , or, which amounts to the same, the d. c. component of the voltage at the output of the ideal detector, connected to the output of the selective system:

$$E_{0.} = M[U] = \int_0^\infty U \frac{U}{\sigma^2} I_0 \left( \frac{U U_0}{\sigma^2} \right) e^{-\frac{U^2+U_0^2}{2\sigma^2}} dU. \quad (6.141)$$

For the computation of the integral (6.141) we shall use the formula

$$\int_0^{\infty} x^2 I_0(qx) e^{-px^2} dx = \frac{\sqrt{\pi}}{4p^{3/2}} e^{q^2/4p} \left\{ I_0\left(\frac{q^2}{8p}\right) + \frac{q^2}{4p} \left[ I_0\left(\frac{q^2}{8p}\right) + I_1\left(\frac{q^2}{8p}\right) \right] \right\}. \quad (6.142)$$

This formula is obtained, if both sides of the equation

$$\int_0^{\infty} I_0(qx) e^{-px^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{p}} e^{-\frac{q^2}{8p}} I_0\left(\frac{q^2}{8p}\right). \quad (6.143)$$

are differentiated with respect to  $p$ .

This is given in the monograph by R. O. Kus'min [41] which deals with the theory of Bessel functions. In view of (6.142) equation (6.141) assumes the following form:

$$E_{0+} = \sqrt{\frac{\pi}{2}} e^{-\frac{U_0^2}{4\sigma^2}} \left\{ I_0\left(\frac{U_0^2}{4\sigma^2}\right) + \frac{U_0^2}{2\sigma^2} \left[ I_0\left(\frac{U_0^2}{4\sigma^2}\right) + I_1\left(\frac{U_0^2}{4\sigma^2}\right) \right] \right\} \quad (6.144)$$

or, in view of (6.42) we obtain

$$\frac{E_{0+}}{E_-} = e^{-\frac{1}{2}\eta^2} \left\{ I_0\left(\frac{\eta^2}{2}\right) + \eta^2 \left[ I_0\left(\frac{\eta^2}{2}\right) + I_1\left(\frac{\eta^2}{2}\right) \right] \right\}. \quad (6.145)$$

where  $\eta = U_0/\sqrt{2}\sigma$  is the ratio between the effective voltages of signal and noise at the output of the selective system.

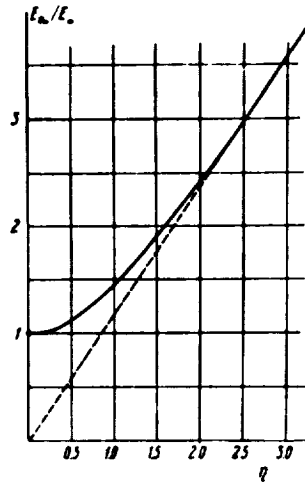


Figure 25. Dependence of the dimensionless rectified voltage on the signal-to-noise ratio at the input of the detector

The graph of the relation  $E_{0+}/E_- = f(\eta)$  corresponding to formula (6.145), is given in Figure 25. It is evident from the graph that a low signal ( $\eta \ll 1$ ) has no substantial influence on the magnitude of the detected voltage, which is essentially determined by the noise. On the other hand, with strong signals ( $\eta \gg 1$ ), the presence of noise has little effect.

Expression (6.138) makes it also possible to calculate the mean square of the resultant amplitude  $U$  and the mean square of its fluctuation component.

### §34. Statistical Properties of the Voltage Phase at the Output of a Selective System under Action of a Nonmodulated Signal and Fluctuation Noise

In the foregoing section the relations (6.136) introduced the random phase  $\theta$  as the angle between the resultant vector  $U$  and the signal vector  $U_0$ . This angle is expressed by the random projections  $u$  and  $v$  as follows,

$$\theta = \operatorname{arctg} \frac{v}{U_0 + u}. \quad (6.146)$$

Let us examine the statistical properties of the phase  $\theta$ . In view of expression (6.34) for the two-dimensional probability density of the projections  $u$  and  $v$  the relation (6.33) for the characteristic function can be given the following form:

$$A(z) = \int_{-\infty}^{+\infty} \int \exp \left[ jz \operatorname{arctg} \frac{v}{U_0 + u} \right] \frac{1}{2\pi\sigma^2} e^{-\frac{u^2+v^2}{2\sigma^2}} du dv. \quad (6.147)$$

By replacing the integration variables in accordance with (6.136) we obtain:

$$A(z) = \int_{-\pi}^{+\pi} e^{jz\theta} d\theta \frac{e^{-\frac{U_0^2}{2\sigma^2}}}{2\pi\sigma^2} \int_0^\infty U e^{-\frac{U^2 - 2UU_0 \cos \theta}{2\sigma^2}} dU. \quad (6.148)$$

For the computation of the inner integral of (6.148) we use the formula

$$\int_0^\infty x e^{-p^2 x^2 + qx} dx = \frac{\sqrt{\pi}}{2p^2} \left\{ \frac{1}{\sqrt{\pi}} + \frac{q}{2p} e^{\frac{q^2}{4p^2}} \left[ 1 + \Phi\left(\frac{q}{2p}\right) \right] \right\}, \quad (6.149)$$

where  $\Phi(z)$  is the error probability integral:

$$\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\frac{x^2}{2}} dx. \quad (6.150)$$

Formula (6.149) can be obtained from the tables /6/

$$\int_0^\infty e^{-p^2 x^2 + qx} dx = \frac{\sqrt{\pi}}{2p} e^{\frac{q^2}{4p^2}} \left[ 1 + \Phi\left(\frac{q}{2p}\right) \right]. \quad (6.151)$$

by differentiating both sides with respect to the parameter  $q$ .

In our case

$$p = \frac{1}{\sqrt{2}\sigma}, \quad q = \frac{U_0}{\sigma^2} \cos \theta, \quad (6.152)$$

and the application of formula (6.149) gives for the probability density of the phase  $\omega(\theta)$  the following expression:

$$w(\theta) = \frac{e^{-\eta^2}}{2\sqrt{\pi}} \left\{ \frac{1}{\sqrt{\pi}} + \eta \cos \theta e^{\eta^2 \cos^2 \theta} [1 + \Phi(\eta \cos \theta)] \right\}. \quad (6.153)$$

As in the foregoing section, we have  $\eta = U_0/\sqrt{2}\sigma$ .

We note that at  $\eta = 0$  (no signal) the expression (6.153) transforms into (6.90).

Figure 26 shows curves of  $w(\theta)$ , constructed from (6.153) for different values of the ratio  $\eta$ . It can be seen from this graph that the increase of the signal amplitude results in a concentration of the most probable values of the phase around  $\theta = 0$ . This means, that under the given conditions the phase of the resulting oscillation is to an ever increasing degree determined by the phase of the signal.

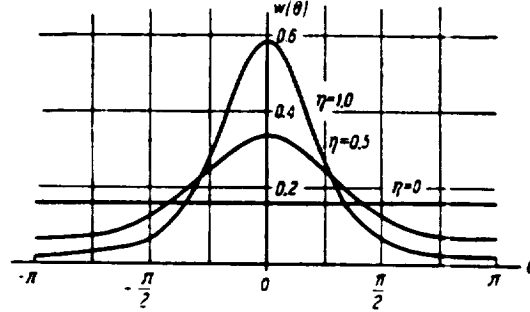


Figure 26. Graphs of the probability density of the phase for different signal-to-noise ratios

### § 35. Computation of the Detector Output for Given Statistical Properties of the Applied Voltage Envelope

It was shown above that the examination of the statistical properties of the voltage envelope at the output of the selective system gave at the same time equivalent results concerning the statistical structure of the voltage at the output of an ideal detector. If the distribution law of the envelope is known, it is easy to calculate also the detection effect for the most varied forms of the volt-ampere characteristics of an inertialess detector. For this we have to know the relation between the mean current value in the detector circuit and the amplitude of the applied alternating voltage. We call this relation the detector characteristic.

We shall consider first a detector with an exponential volt-ampere characteristic, as expressed by the equation (6.1). If a harmonic voltage

$$u = U \cos \omega t = U \cos \alpha, \quad (6.154)$$

is applied to the detector, then the instantaneous current through it becomes

$$i = J_0 e^{\alpha U \cos \alpha}. \quad (6.155)$$

By averaging this current over one cycle of the alternating voltage, we obtain the mean value of the current in the detector circuit:

$$J_{av} = \frac{1}{2\pi} \int_0^{2\pi} i d\alpha = \frac{J_0}{2\pi} \int_0^{2\pi} e^{\alpha U \cos \alpha} d\alpha. \quad (6.156)$$

We apply the formula (6.71) for the calculation of this integral and obtain the equation for the detector characteristic:

$$J_{av} = J_0 \cdot I_0(aU). \quad (6.157)$$

Let the detector be connected to the output of a selective system under fluctuating input. Then the one-dimensional probability density of the voltage amplitude at the detector is by equation (6.39), and the constant component of the current in the detector circuit can be calculated with the aid of (5.20), in which we have to put  $\nu = 1$ :

$$J_{av} = \int_0^{\infty} J_0 \cdot I_0(aU) \cdot \frac{U}{\sigma^2} e^{-\frac{U^2}{2\sigma^2}} dU. \quad (6.158)$$

We can compute this integral by representing the Bessel function  $I_0(aU)$  as a power series (6.15) and then, with the aid of formula (6.44), integrating by parts the thus obtained series in the integrand. Then simple transformations give:

$$J_{av} = J_0 \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{a^2 \sigma^2}{2} \right)^m = J_0 e^{\frac{1}{2} a^2 \sigma^2}, \quad (6.159)$$

i. e., a result coinciding with the expression (6.6).

By analogous methods we can calculate moments of higher order for the mean current, and obtain, for instance, the formula (6.16). Of course, such a calculation method is, in principle, unsuitable for the analysis of the statistical properties of the instantaneous current in a detector circuit, as this current consists not only of the slow-changing mean current but also of rapidly changing high-frequency components. In many cases, however, these latter are of no interest. We shall make use of the same method for the investigation of the statistical properties of the mean current in a square-law detector, whose characteristic is given, as well known, by the equation

$$J_{av} = J_0 + aU^2. \quad (6.160)$$

Using formula (5.20) at  $\nu = 1$  and assuming, as before, that the probability density  $\varphi(U)$  is expressed by (6.38), we obtain:

$$J_{av} = \int_0^{\infty} (J_0 + aU^2) \frac{U}{\sigma^2} e^{-\frac{U^2}{2\sigma^2}} dU. \quad (6.161)$$

Now, the application of formula (6.44) gives

$$J_{av} = J_0 + 2a\sigma^2. \quad (6.162)$$

We note that this result could have been obtained by averaging both sides of equation (6.160) and taking into account that in the case under consideration the mean square of the amplitude  $U$  is determined by the formula (6.45).

By an analogous method it is simple to compute the mean square of the current  $J_{av}$ . Setting in (5.20),  $\nu = 2$ , we obtain:

$$\sigma_{J_{av}}^2 = \int_0^{\infty} (J_0 + aU^2)^2 \frac{U}{\sigma^2} e^{-\frac{U^2}{2\sigma^2}} dU. \quad (6.163)$$

By representing the integral (6.163) as a sum of three integrals and by applying to each of them formula (6.44), we obtain

$$\sigma_{j_{av}}^2 = j_0^2 + 4aj_0\sigma^2 + 8a^2\sigma^4. \quad (6.164)$$

The mean square of the noise component of the mean current is, by virtue of (6.162) and (6.164)

$$\sigma_n^2 = \sigma_{j_{av}}^2 - j_{av}^2 = 4a^2\sigma^4. \quad (6.165)$$

The mixed second moment of the mean current can be calculated, using expression (5.24) and formula (6.75) for the two-dimensional probability density of the envelope. In this case one encounters somewhat greater computational difficulties than those above.

The monograph of V. I. Bunimovich /4/ analyzes beside the problems considered above, a number of other analogous problems. The case of a broken line characteristic of the nonlinear element under normally distributed input is examined in the work by I. N. Amiantov and V. I. Tikhonov /42/.

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